ON THE DESIGN OF L_P IIR FILTERS WITH ARBITRARY FREQUENCY RESPONSE

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ABSTRACT

This paper introduces an iterative algorithm for designing IIR digital filters that minimize a complex approximation error in an L_p sense. The algorithm combines ideas that have proven successful in the similar problem of L_p FIR filter design. We use iterative prefiltering techniques common in applications such as parameter estimation together with an Iterative Reweighted Least Squares (IRLS) method. The result is a double iterative approach that generates IIR filters of arbitrary magnitude and phase response and arbitrary numerator and denominator orders. Such filters can be used in a variety of applications in which the typical L_2 or L_∞ error criteria might not be suitable.

1. INTRODUCTION

Infinite Impulse Response (IIR) filters constitute an important analysis tool in different areas of signal processing. The problem of designing optimal IIR filters has been studied extensively. In practice it is common to design IIR digital filters using techniques such as frequency transformations [1]. Alternatively, one could approximate a frequency response with respect to a specific error norm. In the FIR case one can efficiently design L_2 (least squares) FIR filters by sampling a frequency response and solving a system of normal equations [2]. Furthermore, one can design L_1 or L_∞ FIR filters by posing the design problems as linear programs and using standard techniques such as the Remez exchange algorithm or interior point methods. This flexibility in the design of FIR filters is a consequence of the polynomial approximation problem associated with these filters.

To design an L_p IIR filter given a complex frequency response, one must solve the problem

$$\min_{\hat{a}_i, \hat{b}_i} \|\varepsilon(\omega)\| = \left\| \frac{B(\omega)}{A(\omega)} - D(\omega) \right\|. \tag{1}$$

where

$$B(\omega) = \sum_{n=0}^{M} b(n) e^{-j\omega n}$$

$$A(\omega) = 1 + \sum_{n=1}^{N} a(n) e^{-j\omega n}.$$

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Instead of solving the nonlinear problem (1), one could linearize it as follows

$$\min \|\epsilon(\omega)\| = \|B(\omega) - D(\omega)A(\omega)\|. \tag{2}$$

The notion that (1) and (2) are equivalent is inaccurate. By solving (2), one actually finds the solution to a weighted version of (1), that is

$$\varepsilon(\omega) = W(\omega)\epsilon(\omega).$$

Problem (2) is typically referred as an *equation error* problem, as opposed to the *solution error* problem from (1) in which one is really interested. In the case of least squares approximation, (2) is a weighted least squares problem that is linear in the filter coefficients. This problem is a strictly convex one, and one can solve it by merely equating the gradient of the error to zero and solving for the filter coefficients. This approach was proposed by originally Levy in [3]. Sanathanan and Koerner [4] extended Levy's method into an iterative algorithm to solve the solution error least squares IIR problem (1) given a frequency response. Later, Steiglitz and McBride [5, 6] proposed the same idea for the problem of parameter identification of a system from samples of its impulse response. Although their algorithm was proposed after [4], the Steiglitz-McBride method is far more widely known in the signal processing community. Both algorithms belong to a class commonly referred to as *iterative prefiltering* [7] methods. Alternatively one could also solve (1) using Newton-based methods [8], as would typically be done by an applied mathematician. However, problem (1) is a nonconvex one, and any optimization algorithm aspires to finding at most a local minimizer. Furthermore the problem of finding the Hessian matrix (needed for Newton's method) of the L_p IIR approximation problem in question might not be a realistic goal.

In spite of the discussion above, the problem of designing L_p IIR filters given a frequency response has not been addressed properly. There is no analytical way to minimize the L_p IIR rational approximation error; therefore one must approach this problem iteratively. We propose the use of an Iterative Reweighted Least Squares (IRLS) method to solve (1) in the p-th sense. An IRLS algorithm is one that, at iteration k, solves the weighted least squares problem

$$\min_{\hat{a}_k} \|w(\hat{a}_{k-1}) f(\hat{a}_k)\|_2$$

IRLS methods have been used in geophysics when applied to linear inverse theory problems [9, 10, 11]. The use of IRLS methods in FIR filter design has been introduced in [12, 13]. In [14] an IRLS method was succesfully implemented in the design of L_p FIR filters. In this article we extend the ideas from L_p FIR filters and combine them with iterative prefiltering techniques to propose a two-phase cascaded iterative algorithm to design complex L_p IIR filters. The proposed algorithm has proven to converge efficiently in practice.

2. L_P IIR FILTER DESIGN

In this paper we consider the problem of approximating L arbitrary samples of a complex frequency response function $D(\omega)$ by the IIR filter with M zeros and N poles

$$H(\omega) = \frac{B(\omega)}{A(\omega)} = \frac{\sum_{n=0}^{M} b(n)e^{-j\omega n}}{1+\sum_{n=1}^{N} a(n)e^{-j\omega n}}.$$

The optimization criterion considered is the more general p-norm, with $1 < p < \infty$ (for these values the p-norm is a strict norm, i.e. its unit ball is strictly convex [15]). Therefore the resulting problem is

$$\min_{a,b} \quad \left\| \frac{B(b;\omega)}{A(a;\omega)} - D(\omega) \right\|_p \tag{3}$$

where the p-norm is defined as

$$\|E(\omega)\|_p = \left(\frac{1}{\pi} \int_0^{\pi} |E(\omega)|^p d\omega \right)^{\frac{1}{p}}. \tag{4}$$

In order to solve (3) one must sample $D(\omega)$ and approximate it numerically over the set of sampled frequencies $-\pi \le \omega_i \le \pi$. This leads to the following problem

$$\min \quad e = \|E(\omega)\|_p \tag{5}$$

where

$$\|E(\omega)\|_p = \left(\sum_{i=0}^{L-1} \left| \frac{B(\omega_i)}{A(\omega_i)} - D(\omega_i) \right|^p \right)^{\frac{1}{p}}. \tag{6}$$

Note that (6) involves the p-th root of the error $E(\omega)$, which complicates significantly the optimization problem. Instead, one can consider the equivalent problem $\min \ \varepsilon = e^p = \|E(\omega)\|_p^p$ defined as

$$\min_{a,b} \quad \varepsilon = \|E(\omega)\|_p^p = \sum_{i=0}^{L-1} \left| \frac{B(\omega_i)}{A(\omega_i)} - D(\omega_i) \right|^p \tag{7}$$

since the p-norm of x is a positive monotonic function (both problems are equivalent in the sense that $\min \ \|x\|_p = \min \ \|x\|_p^p$). Therefore a solution of (5) is a set of coefficients a, b (for $a \in \Re^N, b \in \Re^{M+1}$) that minimizes ε in (7). However, we use the fact that problem (7) has better computational properties than problem (5).

3. L_P IIR ALGORITHM

In the previous section we introduced the problem

$$\min_{a,b} \quad \sum_{i=0}^{L-1} \left| \frac{B(\omega_i)}{A(\omega_i)} - D(\omega_i) \right|^p \tag{8}$$

and showed that solving (8) for $\hat{a}, \hat{b}$ is equivalent to solving (4) given a sufficient number of samples of the frequency response function $D(\omega_i)$. In this section we illustrate our approach to (8). Consider

$$\begin{aligned}
\varepsilon &= \sum_{i=0}^{L-1} \left| \frac{B(\omega_i)}{A(\omega_i)} - D(\omega_i) \right|^p \\
&= \sum_{i=0}^{L-1} \left| \frac{B(\omega_i)}{A(\omega_i)} - D(\omega_i) \right|^{p-2} \left| \frac{B(\omega_i)}{A(\omega_i)} - D(\omega_i) \right|^2 \\
&= \sum_{i=0}^{L-1} W(\omega_i) \left| \frac{B(\omega_i)}{A(\omega_i)} - D(\omega_i) \right|^2 .
\end{aligned} \tag{9}$$

Note that (8) is a nonlinear nonconvex problem in the filter coefficients (to see the nonconvexity, consider a filter with one pole). To solve it we proceed with an iterative approach as follows: at the k-th iteration, given coefficient vectors $\hat{a}_k, \hat{b}_k$ one can form an iteration by setting

$$W_k(\omega_i) = \left| \frac{\sum_{n=0}^{M} b_k(n) \exp^{-j\omega_i n}}{1+\sum_{n=1}^{N} a_k(n) \exp^{-j\omega_i n}} - D(\omega_i) \right|^{p-2} \tag{10}$$

and solving for $\hat{a}_{k+1}, \hat{b}_{k+1}$ in (9). One can iterate again by using these vectors in (10) and solving (9) until reaching convergence. This iterative approach has been investigated in [12, 13] and implemented successfully in [14] to the design of linear phase L_p FIR filters.

The approach mentioned above requires the solution of the nonlinear weighted least squares problem (9) at each iteration. To solve it, one can linearize the system (9) by prefiltering it with a filter corresponding to $\frac{1}{A(\omega_i)}$. Consider

$$\begin{aligned}
\varepsilon &= \sum_{i=0}^{L-1} W_k(\omega_i) \left| \frac{B(\omega_i)}{A(\omega_i)} - D(\omega_i) \right|^2 \\
&= \sum_{i=0}^{L-1} \frac{W_k(\omega_i)}{|A(\omega_i)|^2} |B(\omega_i) - A(\omega_i)D(\omega_i)|^2 .
\end{aligned} \tag{11}$$

One can solve (11) for $\hat{a}_{l+1}, \hat{b}_{l+1}$ iteratively by letting

$$\hat{W}_l(\omega_i) = \frac{W_k(\omega_i)}{|A_l(\omega_i)|^2} \tag{12}$$

at the l-th iteration. Therefore by fixing $A_l(\omega_i)$ in (12) one can write (11) as

$$\varepsilon = \sum_{i=0}^{L-1} \hat{W}_l(\omega_i) |B_{l+1}(\omega_i) - A_{l+1}(\omega_i)D(\omega_i)|^2 .$$

Since $\hat{W}_l(\omega_i)$ is fixed, problem (11) corresponds to a weighted linear least squares problem. An important property of such system is that it is indeed strictly convex in the coefficients, with a unique solution. Therefore it is only necessary to equate the gradient of ε with respect to the filter coefficients to zero and solve for the coefficients,

$$\nabla_{\hat{a},\hat{b}} \varepsilon = \mathbf{0}. \tag{13}$$

A detailed explanation of this step is discussed in [3], and is typically referred as Levy's method [4].

From the discussion above it is clear that one can solve the L_p IIR problem (7) iteratively. One must also note that during each

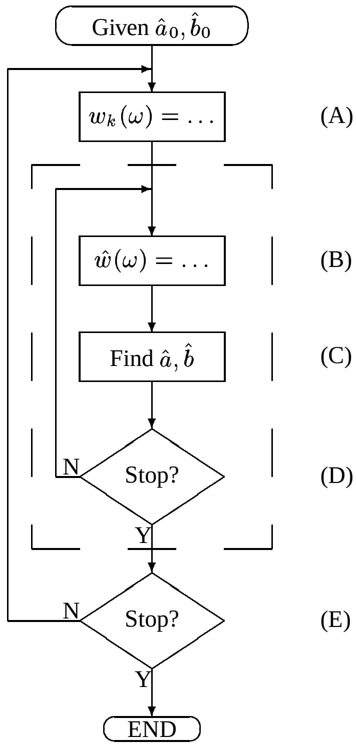

Fig. 1. *Block diagram for the iterative algorithm.*

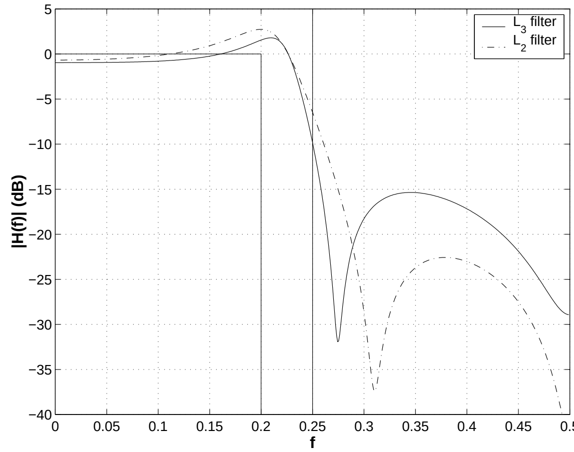

Fig. 2. *Comparison of magnitude responses for L_2 and L_3 IIR filters.*

iteration of the L_p problem, an internal simpler iterative method is performed. This concatenation of iterative methods can be better understood by referring to Figure 1.

Basically the algorithm proposed is: Given $\hat{a}_0, \hat{b}_0$,

1. Let
$$w_k(\omega) = \left| \frac{B(\omega)}{A(\omega)} - D(\omega) \right|^{p-2}.$$
2. Find $\hat{a}_{k+1}$ and $\hat{b}_{k+1}$:
   (a) Let
   $$\hat{w}(\omega) = \frac{w_k(\omega)}{|A(\omega)|^2}.$$
   (b) Find $\hat{a}_{k+1}$ and $\hat{b}_{k+1}$ from
   $$\min \sum_{i=0}^{L-1} \hat{w}(\omega_i) |B(\omega_i) - D(\omega_i) A(\omega_i)|^2.$$
   (c) Check for convergence (D).
3. Check for convergence (E).

Step 2.b is the weighted linear least squares problem from (11).

4. RESULTS

The proposed algorithm was implemented to design L_p IIR filters given a specified frequency response. Figure 2 shows the frequency response of a filter with 3 zeros and two poles, as compared to the L_2 filter. The desired frequency response is a zero-phase lowpass filter with passband frequency 0.2 and stopband frequency 0.25 (normalized frequency). The filter is optimal in the L_3 sense. To illustrate the comparison between both filters we found the following errors

$$\|e_2(\omega)\|_3 = 5.129, \qquad \|e_3(\omega)\|_3 = 4.649.$$

Figure 3 shows the frequency response for a filter with 5 zeros and two poles. Again, the desired frequency response was a zero-phase lowpass filter with passband frequency 0.2 and stopband frequency 0.22. The approximation was done using the L_4 norm. The results obtained were

$$\|e_2(\omega)\|_4 = 4.565, \qquad \|e_4(\omega)\|_4 = 4.337.$$

The algorithm is split into two parts: on the outer iteration one performs an IRLS update. Typically less than 5 to 10 iterations are needed before convergence. At each outside iteration an internal iterative procedure is performed. Convergence is usually achieved after 5 to 20 iterations. The total algorithm usually converges in a few seconds in a 450MHz Pentium III PC running Windows NT 4.

Unlike the Butterworth, Chebyshev and Cauer analytical methods, our approach allows using a small number of poles and a large number of zeros in a filter. Typically we saw that an adequate rule for the number of poles is to assign one pole per transition band (i.e. 2 poles for a lowpass filter). Only a few poles are needed for a narrow transition band; as many zeros as needed for small ripple can be used.

5. CONCLUSIONS

In this paper we presented a new algorithm to solve the problem of minimizing the p-norm of the L_p approximation error for IIR filters given a complex frequency response. The proposed algorithm is a double iterative procedure. The outside part performs an Iterative Reweighted Least Squares method to solve a nonlinear weighted least squares problem. At each iteration of this process, an internal iterative procedure (based on iterative prefiltering) is used to solve a linear weighted least squares problem.

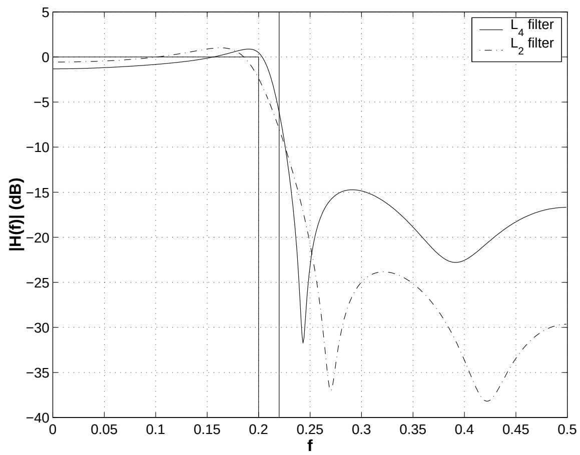

Fig. 3. *Comparison of magnitude responses for L_2 and L_4 IIR filters.*

We presented results on filters that are locally optimal in the L_p sense. Starting from an equation error solution, our algorithm moves through the nonconvex error surface to find a local minimizer. Further work must be done to combine this algorithm with other optimization techniques for global minimization. Our experiments show that for an adequate order selection, our algorithm converges rather rapidly. Further analysis of convergence properties are still to be done. However the methods used to derive our algorithm have been shown to adequately converge in practice [14]. This suggests the adequate convergence properties of our approach.

In the past, the problem of designing L_p IIR filters has not been addressed as it should. We proposed an algorithm that can further be modified to accomodate different error criteria in different frequency bands. This idea could prove useful as an alternative to designing constrained least squares filters.

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