ADAPTIVE CFAR DETECTION OF MULTIDIMENSIONAL SIGNALS

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ABSTRACT

Adaptive detection of multidimensional signals in the presence of interference with unknown covariance matrix is an expanding topic in a variety of scenarios ranging from radar/sonar to digital communication systems. In this paper we attack the problem of detecting a multidimensional radar signal, modeled as an unknown $N \times H$ matrix, embedded in Gaussian noise with unknown covariance matrix, with the ambition of devising receivers which yield the Constant False Alarm Rate (CFAR) property. We show that this aim can be achieved resorting to the principle of invariance, namely restricting our attention to hypothesis testing problems which remain unaltered under a proper group of transformations. Several detectors based on the maximal invariant statistic are studied and, in particular, the Generalized Likelihood Ratio Test (GLRT) is shown to belong to the class of invariant tests.

1. INTRODUCTION

Adaptive detection of a signal array in a background of interference is a major point in the design of radar/sonar and digital communication systems. A specific problem of primary concern is the detection of a signature known up to a multiplicative constant, in the presence of zero-mean, Gaussian noise with unknown covariance matrix. It has been addressed by several investigators and a considerable bulk of work is available, as for instance Kelly's GLRT [1], and the Adaptive Matched Filter [2].

hypothesis testing problems concerning the presence of multidimensional signals, i.e. signals known to be confined to a given vector subspace, have also been widely studied. Among statisticians such hypothesis tests for a structured mean in multivariate Gaussian noise with unknown covariance are usually referred to as Generalized Multivariate Analysis of Variance (GMANOVA) [3]. More recently, the GMANOVA has also been introduced in the signal processing literature by Kelly and Forsythe [4][and references therein]. In [4] the authors model the useful signal to be detected as the product of three matrices, namely as $\boldsymbol{\sigma} \mathbf{B} \boldsymbol{\tau}$, where $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are fixed arrays accounting for the structure of the useful signal, while $\mathbf{B}$ is an array of unknown signal amplitude parameters. The above model is quite general and encompasses the detection of either coherent and incoherent point-like or range-spread targets as viewed through an array of sensors of some kind; in addition, it allows to include among data additional returns free of signal components and with the same unknown covariance matrix of the cells under test.

A desirable feature of an adaptive detection scheme is the CFAR property, meaning that the threshold value, necessary to set the desired false alarm Probability (P_{fa}), is independent of the unknown covariance matrix of the disturbance. It is possible to impose the CFAR constraint resorting to the principle of *invariance* [5, 6]. The core of the procedure is to define a class of tests which are invariant with respect to a group of transformations that leave the hypothesis testing problem unaltered. It is worth pointing out that any invariant decision rule can be expressed in terms of a vector valued function of the data, the so-called *maximal invariant*. Thus, knowledge of the maximal invariant greatly reduces the overall class of detectors to be considered. It is also possible that a Uniformly Most Powerful (UMP) test exists within the class of invariant tests, when no general UMP test may exist.

Several researchers resorted to the principle of invariance to address adaptive CFAR detection. For instance, the detection of a subspace signal in subspace interference plus broadband noise is addressed in [7]. The problem of detecting a known (within a complex scale factor) vector in structured interference with unknown correlation properties is considered, instead, in [8]. Moreover, the application of the invariance to GMANOVA problems can also be found in [9] and in [10]. The former refers to the detection of a completely unknown N-dimensional vector, whereas the latter addresses the case of a signal confined to a given vector subspace. Both of them assume to estimate the correlation properties of the disturbance from a set of secondary data. Finally, in [11] the invariance principle is exploited to develop a framework for adaptive array detection of uncertain rank-one waveforms.

In this paper we address adaptive detection for multidimensional signals imposing the CFAR constraint. Again we resort to the principle of invariance. More precisely, we assume that the data set consists of H vectors, which may contain returns from possible target or targets, plus K additional vectors, free of signal components and with the same unknown covariance matrix of the cells under test. The signatures of the possible useful signals are not known at the receiver; it follows that the above model applies to the detection of incoherent, range-spread targets as well as to multispectral or multipolarization images collected by a synthetic aperture radar. In the latter case the columns of the matrix of primary data correspond to target returns at different frequency bands (multispectral images) or at different polarizations (multipolarization images) while the rows correspond to different spatial locations (pixels) [12]. In addition, we provide the set of parameters which rule the performance of all the invariant detectors (induced maximal invariant). It is shown that no UMP Invariant (UMPI) test for the given detection problem exists. Subsequently, we introduce and assess several maximal invariant based decision rules. In particular, we show that the Kelly-Forsythe GLRT is a member of the class of invariant tests.

The outline of the paper is the following: Section 2 is devoted to the problem formulation and to the derivation of the maximal invariant; Section 3 addresses the design of maximal invariant based detectors. The performance assessment is reported in Section 4 while concluding remarks are presented in Section 5.

2. PROBLEM FORMULATION AND DESIGN ISSUES

We assume that data are collected from N sensors and deal with the problem of detecting the presence of a signal within H data vectors, $\boldsymbol{z}_t$, $t = 1, \ldots, H$, referred to, in the sequel, as primary data. We also suppose that a secondary data set, $\boldsymbol{z}_t$, $t = H + 1, \ldots, H+K$, free of useful signal components and which exhibits the same covariance matrix as the primary data, is available.

The detection problem to be solved can be formulated in terms of the following binary hypothesis test:

$$\begin{cases} H_0: \quad \boldsymbol{z}_t = \mathbf{n}_t, \quad t = 1, \ldots, H+K \\ \\ H_1: \begin{cases} \boldsymbol{z}_t = \boldsymbol{p}_t + \mathbf{n}_t, & t = 1, \ldots, H \\ \boldsymbol{z}_t = \mathbf{n}_t, & t = H+1, \ldots, H+K \end{cases} \end{cases} \tag{1}$$

where the $\boldsymbol{p}_t$'s, $t = 1, \ldots, H$, are unknown vectors, and the $\mathbf{n}_t$'s $t = 1, \ldots, H + K$, are independent, zero-mean Gaussian vectors with covariance matrices given by

$$E[\mathbf{n}_t \mathbf{n}_t^\dagger] = \mathbf{M}, \quad t = 1, \ldots, H+K\,. \tag{2}$$

As to $E[\cdot]$ it denotes statistical expectation and † is the conjugate transpose. Moreover, we suppose that the $\mathbf{n}_t$'s possess the circular property usually associated with I and Q pairs of a Wide-Sense Stationary process. The vectors $\boldsymbol{z}_t$, $t = 1, \ldots, H + K$, can be organized into the matrices $\boldsymbol{Z}_p = [\, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_H]$ and $\boldsymbol{Z}_s = [\, \boldsymbol{z}_{H+1}, \ldots, \boldsymbol{z}_{H+K}]$, referred to, in the sequel, as primary and secondary data matrices, respectively.

We preliminary notice that the hypothesis testing problem (1) is invariant [5, 6] under the group of transformations $\mathbf{G}$ defined as:

$$\mathbf{G}: \quad \boldsymbol{Z} = [\, \boldsymbol{Z}_p,\ \boldsymbol{Z}_s] \to \boldsymbol{B}\,\boldsymbol{Z}\,; \tag{3}$$

where $\boldsymbol{B}$ denotes a nonsingular $N \times N$, possibly complex matrix. In fact $\mathbf{G}$ maps independent and identically distributed (iid) Gaussian vectors with unknown covariance matrix into iid Gaussian vectors with unknown, but different, covariance matrix. Moreover, if a data vector is zero-mean the transformed one will remain zero-mean.

In the sequel we focus on test statistics which are invariant under $\mathbf{G}$, namely, if we apply the decision rules to the transformed data matrices we obtain the same answer as if the tests had been applied to the original data. As a consequence the class of decision rules we consider ensures CFARness with respect to the covariance matrix of the disturbance. In fact, it can be easily proved that, under H_0, the distribution of any $\mathbf{G}$-invariant test statistic is independent of $\mathbf{M}$.

It is a remarkable result [5, 6] that all invariant tests can be expressed as a function of a maximal invariant statistic which acts on the measurements organizing them into orbits or equivalence classes. For reader's ease we remind that a statistic $\mathbf{T}$ is said to be a maximal invariant for the group of transformations $\mathbf{G}$ if and only if

- $\mathbf{T}(\,\boldsymbol{Z}) = \mathbf{T}(\,\boldsymbol{B}\,\boldsymbol{Z}), \qquad \forall\, \boldsymbol{B} \in \mathbf{G};$
- $\mathbf{T}(\,\boldsymbol{Z}^{'}) = \mathbf{T}(\,\boldsymbol{Z}^{''}) \Longrightarrow \boldsymbol{Z}^{'} = \boldsymbol{B}\,\boldsymbol{Z}^{''}$ for some $\boldsymbol{B} \in \mathbf{G}$.

The following proposition specifies a maximal invariant statistic for the problem at hand.

Proposition 1. Assume $K \geq N$, then a maximal invariant statistic for the group of transformations (3) is given by the $r = min(H, N)$ non-zero eigenvalues, $\lambda_1, \ldots, \lambda_r$ say, of the Hermitian matrix

$$\boldsymbol{S}^{-\frac{1}{2}}\,\boldsymbol{\Sigma}\,\boldsymbol{S}^{-\frac{1}{2}}\,, \tag{4}$$

where

- $\boldsymbol{S} = \boldsymbol{Z}_s\,\boldsymbol{Z}_s^\dagger$ is K times the sample covariance matrix of the secondary data;
- $\boldsymbol{\Sigma} = \boldsymbol{Z}_p\,\boldsymbol{Z}_p^\dagger$ is H times the sample covariance matrix of the primary data.

The results of the above proposition require some comments. First of all we point out that the maximal invariant statistic has a precise geometrical interpretation; in fact, its elements represent the lengths of the semi-axes of the hyperhellipsoid $\mathcal{E}$ defined as

$$\mathcal{E} = \{\, \boldsymbol{S}^{-\frac{1}{2}}\,\boldsymbol{\Sigma}\,\boldsymbol{S}^{-\frac{1}{2}}\mathbf{x}, \qquad \|\mathbf{x}\| = 1\}$$

where, in turn, $\|\mathbf{x}\|$ denotes the Euclidean norm of a complex vector [13][pp. 72]. Moreover, we explicitly note that the maximal invariant statistic for the problem at hand is r-dimensional. Thus, if $r > 1$, no UMPI test exists [6]. On the contrary, when $H = 1$, the rank of the matrix $\boldsymbol{S}^{-\frac{1}{2}}\,\boldsymbol{\Sigma}\,\boldsymbol{S}^{-\frac{1}{2}}$ reduces to one and, as a consequence, the maximal invariant turns out to be the non-zero eigenvalue

$$\lambda_1 = tr(\,\boldsymbol{S}^{-\frac{1}{2}}\,\boldsymbol{\Sigma}\,\boldsymbol{S}^{-\frac{1}{2}}) = \boldsymbol{z}_1^\dagger\,\boldsymbol{S}^{-1}\,\boldsymbol{z}_1\,,$$

where $tr(\cdot)$ denotes the trace of a square matrix. In this specific case, the statistic coincides with the one proposed in [9] which has also been shown to be UMPI.

As to the statistical characterization of the maximal invariant, we note that it is possible to determine the joint probability density function of the ordered $\lambda_1, \ldots, \lambda_r$, either under H_0 and H_1. Moreover, it can be shown that the distribution under H_1 depends only upon the reduced set of parameters $\boldsymbol{\Omega} = (\omega_1, \ldots, \omega_N)$ (induced maximal invariant), where ω_i, $i = 1, \ldots, N$, denote the roots of the determinantal equation $|\,\boldsymbol{P}\,\boldsymbol{P}^\dagger - \omega\mathbf{M}| = 0$, and $\boldsymbol{P} = [\,\boldsymbol{p}_1, \ldots, \boldsymbol{p}_H]$.

Finally, we observe that the amount of work required for the evaluation of the maximal invariant is $O(HN^2) + O(KN^2)$ floating point operations (flops)[1].

[1] Herein we use the usual Landau notation $O(n)$; hence, an algorithm is $O(n)$ if its implementation requires a number of flops proportional to n [13].

3. MAXIMAL INVARIANT BASED DECISION RULES

The lack of an UMPI test for the most general case of $r > 1$ suggests to investigate invariant decision rules based upon several criteria. First we focus on two different GLRT-based design procedures since they often lead to invariant detectors. More precisely, we consider one-step and two-step GLRT strategies. The former, proposed by Kelly and Forsythe [4], is tantamount to replace the unknown parameters in the conditional likelihood ratio with their maximum likelihood estimates, under each hypothesis, based upon the entirety of data [14]. The latter, instead, first assumes that the covariance matrix $\mathbf{M}$ is known and derive the GLRT maximizing the Likelihood Ratio over the unknown signatures of the signal (step 1). Then, after the GLRT is derived, the sample covariance matrix based upon secondary data is inserted, in place of the true covariance matrix, into the test (step 2). We highlight that this procedure leads, in general, to simpler decision rules, which, as shown in [2] (for point-like target and coherent detection) and in [15] (for the case of distributed targets and coherent detection), may achieve higher detection probabilities than the one-step GLRT. It can be shown that the plain GLRT and the two-step GLRT-based design procedure lead to

$$\prod_{t=1}^{r}(1+\lambda_t) \underset{H_0}{\overset{H_1}{\gtrless}} T \tag{5}$$

and

$$\sum_{t=1}^{r}\lambda_t \underset{H_0}{\overset{H_1}{\gtrless}} T\,, \tag{6}$$

respectively, and, hence, they turn out to be invariant.

We also deal with another two maximal invariant detectors, known in the statistical literature as the spectral norm test and the Pillai-Bartlett trace test [6], i.e.

$$\max_{t=1,\ldots,r}\lambda_t \underset{H_0}{\overset{H_1}{\gtrless}} T \tag{7}$$

and

$$\sum_{t=1}^{r}\frac{\lambda_t}{1+\lambda_t} \underset{H_0}{\overset{H_1}{\gtrless}} T\,, \tag{8}$$

respectively. It can be shown that the latter can be derived resorting to a modified two-step GLRT design procedure where the step 1 coincides with that of the conventional two-step procedure. The second step, instead, relies upon the estimation of the covariance matrix $\mathbf{M}$ based upon the entirety of data and assuming that the hypothesis H_0 is in force.

Finally, we observe that, if a UMP invariant test exists, i.e. $H = 1$, all of the above $\mathbf{G}$-invariant decision rules collapse into the same test, namely the one based on the non-zero eigenvalue of $\boldsymbol{S}^{-\frac{1}{2}}\,\boldsymbol{\Sigma}\,\boldsymbol{S}^{-\frac{1}{2}}$, which, as mentioned before, is UMP invariant.

4. PERFORMANCE ASSESSMENT

In this section we discuss the performance of the maximal invariant based receivers introduced in Section 3. First of all it can be shown that the tests (5), (6), and (8) are asymptotically equivalent when the number of secondary data K becomes increasingly large. Moreover, the asymptotic P_{fa} and Probability of detection (P_d) coincide with those of the GLRT for known covariance matrix, and are given by

$$P_{fa} = e^{-\frac{T}{2}}\sum_{j=0}^{NH-1}\frac{1}{j!}\left(\frac{T}{2}\right)^j\,, \tag{9}$$

and

$$P_d = \mathcal{Q}_{NH}\left(\sqrt{\sum_{t=1}^{H}\boldsymbol{p}_t^{\dagger}\mathbf{M}^{-1}\,\boldsymbol{p}_t},\sqrt{T}\right)\,, \tag{10}$$

where $\mathcal{Q}_{NH}(\cdot,\cdot)$ denotes the Marcum Q-function.

However, for finite K, the performance of the detectors (5), (6), (7), and (8) depend upon the number of available secondary data. It is thus of interest to study the behavior of P_{fa} and P_d for finite values of K. To this end we resort to numerical simulations based upon the Monte Carlo counting technique. More precisely, P_{fa} and P_d are estimated using $\frac{100}{P_{fa}}$ and $\frac{100}{P_d}$ independent trails respectively, and in order to limit the computational burden we fix $P_{fa} = 10^{-4}$. The disturbance covariance matrix, we consider, is exponentially-shaped, i.e.

$$\mathbf{M} = \parallel m_{i,j} \parallel = 2\sigma^2 \parallel \rho^{|i-j|} \parallel, \qquad \rho > 0$$

where σ^2 is the common variance of the noise quadrature components and ρ is the one-lag correlation coefficient. As to the useful signal, instead, we suppose that $\boldsymbol{p}_t$'s, $t = 1,\ldots,H$ are independent, zero-mean, complex Gaussian vectors with covariance matrix $\mathbf{M}_{TG} = \gamma_t\boldsymbol{\Sigma}_{TG}$, where the γ_t's account for the distribution of the signal energy among the primary data and the normalized covariance $\boldsymbol{\Sigma}_{TG}$ is exponentially-shaped with one-lag correlation coefficient ρ_{TG}. In Figure 1 the P_d of the proposed detectors is reported versus the average SNR, i.e.

$$SNR = E\left[\sum_{t=1}^{H}\boldsymbol{p}_t^{\dagger}\mathbf{M}^{-1}\,\boldsymbol{p}_t\right] = tr(\boldsymbol{\Sigma}_{TG}\mathbf{M}^{-1})\sum_{t=1}^{H}\gamma_t\,,$$

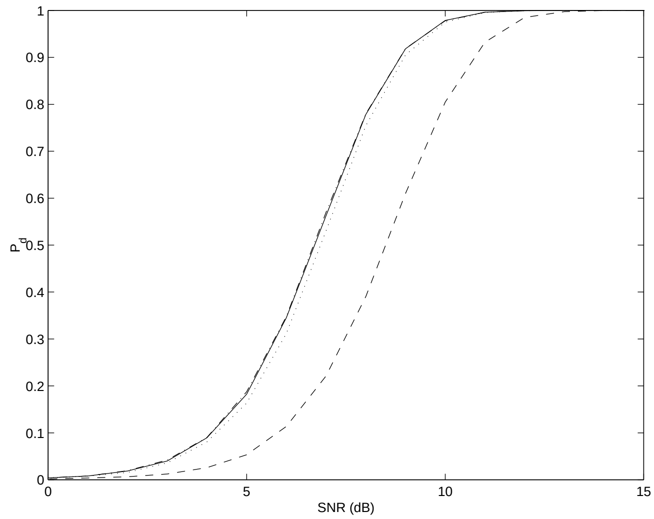

Figure 1: P_d versus SNR of the detectors (5) (solid curve), (6) (dotted curve), (7) (dashed curve), and (8) (dash-dot curve), for $H = 4$, $N = 8$, $K = 48$, $\rho = 0.9$, $\rho_{TG} = 0.6$, $P_{fa} = 10^{-4}$, and signal model 1.

Signal Model	Data Number			
	1	2	3	4
1	1/4	1/4	1/4	1/4
2	1	0	0	0

Table I: distribution of signal energy, $\dfrac{\gamma_t}{\sum_{t=1}^{H} \gamma_t}$, $t = 1, \ldots, H$.

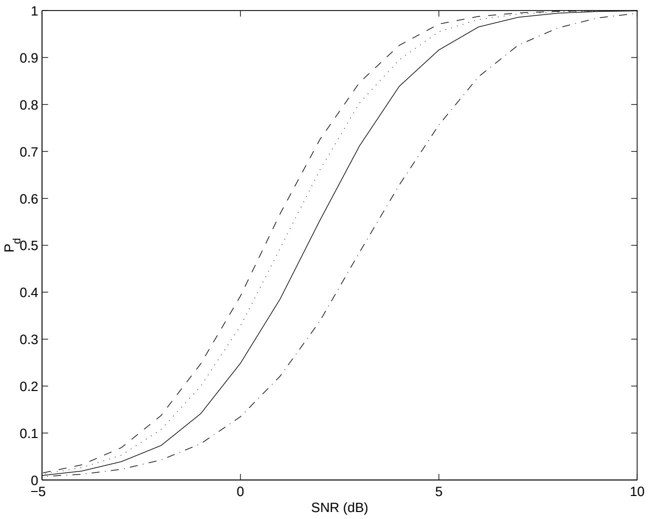

Figure 2: P_d versus SNR of the detectors (5) (solid curve), (6) (dotted curve), (7) (dashed curve), and (8) (dash-dot curve), for $H = 4$, $N = 8$, $K = 48$, $\rho = 0.9$, $\rho_{TG} = 0.6$, $P_{fa} = 10^{-4}$, and signal model 2.

for $P_{fa} = 10^{-4}$, $H = 4$, $N = 8$, $K = 48$, $\rho = 0.9$, $\rho_{TG} = 0.6$, and signal model 1 (see Table I). The figure shows that, for the parameters value chosen, detectors (5), (6), and (8) achieve almost the same performance and outperform the receiver (7). Figure 2, instead, refers to $P_{fa} = 10^{-4}$, $H = 4$, $N = 8$, $K = 48$, $\rho = 0.9$, $\rho_{TG} = 0.6$, and signal model 2. In this case the spectral norm test performs better than the other detectors. Moreover, inspection of the figures and simulation results not reported here for the lack of space, clearly highlight that the performance depends upon the signal model being in force.

5. CONCLUSIONS

In this paper we have addressed the detection of a multidimensional signal in a background of Gaussian noise with unknown covariance. In order to come up with fully CFAR receivers we resorted to the principle of invariance. Since no UMPI test exists, we have introduced and assessed several maximal invariant based receivers. Simulation results have highlighted that the performances depend upon the distribution of the signal energy among the primary data, but, for a given signal configuration, the loss between the different detectors is no more than about 3 dB. From a preliminary analysis of a wider class of results, collected for various configurations of primary and secondary data, we observe that the GLRT receiver performs better than the others in the case of uniform distribution of the signal energy. Conversely, in the case of concentrated energy, the spectral norm test guarantees the highest performance. In conclusion, none of the receivers is uniformly better than the others, but some knowledge about the distribution of the signal energy allows a proper selection among the proposed detectors.

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