LOW-COMPLEXITY TRANSFORM CODING WITH INTEGER-TO-INTEGER TRANSFORMS

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ABSTRACT

In this paper we propose the application of a new transform-based coding method[1] in conjunction with Golomb-Rice (GR) codes to lower significantly the complexity, which can be used in various applications, e.g. the *Multiple Description* coding[2]. The theoretical evaluations predict no important loss in compression performance, while the complexity is considerably reduced. Since GR codes are very fast and well suited for exponentially decaying distributions, they were implemented during the last decade in image and audio compressors. In all these schemes, the selection of the code parameter is performed presuming Laplacian distribution of prediction errors. We derive the selection method for the GR code parameter also for the case of Gaussian inputs.

1. INTRODUCTION

The *Multiple Description* (MD) coding became a popular topic of research in the recent years because when is applied to the design of communication systems makes them robust to packet losses. One practical embodiment of MD paradigm is the implementation of PAC coder in [2]. We are interested in this paper in one essential block in the referred coding scheme, namely the integer-to-integer transform followed by entropy coding. On a different thread, the integer-to-integer transform coding (I_2ITC) was proposed as an alternative to transform coding[1], where the customary transform–quantization tandem is reverted such that the quantizer is followed by the transform, and finally the transformed vector is entropy coded.

The specific steps of I_2ITC in the case of a zero-mean jointly Gaussian two-dimensional source vector x are: (1) uniform quantization, (2) integer-to-integer invertible mapping, and (3) separately encoding of the resulting descriptions with Huffman codes. The integer-to-integer transform is optimized in order to minimize the divergence between the distributions of the two outputs. Similar distributions of the transform outputs are in average more favorable under packet losses than the case of outputs with differently skewed distributions. Moreover, since the outputs have similar distributions, the same Huffman tables may be used for both channels, which leads to a reduction of the algorithm complexity[1].

We propose a further step towards the complexity reduction of I_2ITC by replacing Huffman coding with *Golomb-Rice coding* (GR). We will show that *Golomb-Rice coding* is well suited for processing the output of the integer-to-integer transform in the case of Gaussian input, and we will study the problem of GR code parameter selection. The *Golomb codes* were introduced in [3] to encode arbitrary non-negative integers with *one-sided geometric distribution* (OSG): $P(n) = (1-q)q^n$, $q \in (0,1)$. The optimality of the proposed codes interpreted as Huffman codes for infinite alphabets was proven and recently *Golomb codes* were successfully used for coding *two-sided geometric sources* (TSG).

The *Golomb-Rice codes* with parameter k, denoted GR_k, encodes a nonnegative integer u by sending the least significant k bits of u, followed by the unary representation of $u_k = \lfloor \frac{u}{2^k} \rfloor$ (a sequence of u_k ones), and finally terminates the codeword with a 0 bit to allow unique decoding. This results in a total number of bits $u_k + k + 1$. The GR coding is the low-complexity method of choice in research as well as in practice, being used to encode prediction residuals x which are supposed to be Laplacian distributed.

We will investigate in Section 3 the selection of the code parameter in the hypothesis of Gaussian distribution for samples x. First the samples x are quantized to integer-valued $\hat{x}$. One possible choice for coding $\hat{x}$[4] is to encode separately the sign and the modulus $\tilde{x} = |\hat{x}|$. This is the procedure we will also consider in the sequel. For simplicity, we assume the average code length for the sign is 1 bit and consider only the coding of modulus $\tilde{x}$.

Related results concerning the code redundancy and efficiency have been presented in [5], which addresses the problem of GR encoding the quantized generalized Gaussian pdf's, but their methods for selecting the code parameter have higher memory requirements, involving the storage of tables designed off-line. Another related paper is [4]: the GR code parameter k is selected to obey the condition

$$\int_{-2^k}^{2^k} f(x|\sigma)dx = 1/2 \tag{1}$$

where $f(x|\sigma)$ is the Laplacian pdf. A similar technique will be presented in this paper to select the code parameter for Gaussian inputs.

2. TRANSFORM CODING WITH INTEGER-TO-INTEGER TRANSFORMS

Suppose the input vector $\underline{x}(t) = [x_1(t), x_2(t)]^T \in \mathcal{R}^2$ is Gaussian distributed, having the pdf

$$f(\underline{x}) = \frac{1}{2\pi\sqrt{|K|}} \exp\left(-\frac{1}{2}\underline{x}^T K^{-1} \underline{x}\right)$$

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where K is the covariance matrix $K = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$, $\rho^2 < 1$, $\sigma_1, \sigma_2 > 0$ and $|K| = \sigma_1^2\sigma_2^2\left(1-\rho^2\right)$ is the determinant of K. We assume $\underline{x}$ is a zero mean vector, $E[\underline{x}] = \underline{0}$.

2.1. Quantization of the input

Let consider an uniform quantizer $Q_{N,\Delta}$ with $N = 2^{M+1} - 1$ (M is a positive integer) reconstruction levels and step size Δ, such that the quantized version $\hat{x}$ of the real-valued variable x is given by:

$$Q_{N,\Delta}(x) = \begin{cases} -\frac{N-1}{2}\Delta, & x < -\left(\frac{N}{2}-1\right)\Delta \\ i\Delta, & \left(i-\frac{1}{2}\right)\Delta \le x < \left(i+\frac{1}{2}\right)\Delta, \\ & -\frac{N-1}{2} < i < \frac{N-1}{2} \\ \frac{N-1}{2}\Delta, & x \ge \left(\frac{N}{2}-1\right)\Delta \end{cases} \tag{2}$$

For N large ($N \to \infty$), the quantizer $Q_{N,\Delta}$ applied to the r.v. x generates the r.v. $\hat{x}$ given by:

$$\hat{x} = Q_{N,\Delta}(x) = i\Delta \text{ when } \left(i-\frac{1}{2}\right)\Delta \le x < \left(i+\frac{1}{2}\right)\Delta \tag{3}$$

for any integer i.

In the sequel we are briefly discussing the quantization of $\underline{x}$ by using different approaches:

Direct quantization (DQ): We use $Q_{N,\Delta}$ to quantize $x_1(t)$ and $x_2(t)$ obtaining $\hat{x}_1(t) = i_1\Delta$ and $\hat{x}_2(t) = i_2\Delta$. The integers $[i_1\ i_2]$ have the joint probability mass function (pmf) $P(i_1, i_2) = \int_{x_1\in V_{i_1}}\int_{x_2\in V_{i_2}} p([x_1, x_2])dx_1dx_2$ where V_{i_1} and V_{i_2} denotes the Voronoi cells of the reconstructions $i_1\Delta$ and $i_2\Delta$.

Unitary transform coding (UTC): The Karhunen-Loeve transform (KLT) U is first applied to $\underline{x}$ resulting in $\underline{y} = U\underline{x}$. Then $Q_{N,\Delta}$ is applied to the entries of $\underline{y}$ to obtain $\hat{\underline{y}}$. When $\underline{x}$ is jointly Gaussian distributed, the KLT ensures that the components of y are statistically independent and $\hat{y}_1$, $\hat{y}_2$ can be efficiently encoded with scalar, but in general different, entropic coders.

Integer-to-integer transform coding ($\mathbf{I_2ITC}$)[1]: The indexes $[i_1\ i_2]$ obtained after quantization as in method DQ are mapped to $[i_1'\ i_2']$ with a reversible integer transform $\hat{T}$. The new obtained indexes are encoded. The main difference with respect to UTC method consists in using a discrete transform instead of a continue transform.

The distortion for the uniform scalar quantizer $Q_{N,\Delta}$ is given by $D \approx \Delta^2/12$ in the hypothesis of high-resolution approximation (Δ small). When the components of the quantized vector $[\hat{x}_1\ \hat{x}_2]$ are not independent, the per-component distortion is computed with the expression $\frac{1}{2}||\underline{x} - \hat{\underline{x}}||^2$. In [1] it is shown that the distortion is nearly the same for all the methods (DQ,UTC,I_2ITC) when Δ is small. So, the distortion is fixed, but the rate is variable and the design we describe has the goal to minimize the rate.

For small Δ we may assume the pdf of $\underline{x}$ to be constant over a cell $V_{i_1} \times V_{i_2}$, and therefore the joint pmf can be approximated as

$$P([\hat{x}_1\ \hat{x}_2]^T) \approx Ce^{-\frac{[\hat{x}_1\ \hat{x}_2]K^{-1}[\hat{x}_1\ \hat{x}_2]^T}{2}} \tag{4}$$

with C a normalization constant. It results that the probability for $[i_1\ i_2]^T$ can be approximated as follows:

$$P([i_1\ i_2]^T) \approx Ce^{-\frac{\Delta^2[i_1\ i_2]K^{-1}[i_1\ i_2]^T}{2}} \tag{5}$$

In the rest of the paper we use equality sign instead of approximation sign each time we are referring to the pmf of $[\hat{x}_1\ \hat{x}_2]^T$ or $[i_1\ i_2]^T$. Due to the symmetry of the discrete distribution, $E[\hat{x}_1\ \hat{x}_2]^T = E[i_1\ i_2]^T = \underline{0}^T$. After some calculations on the moments for the distribution of the random vector $[\hat{x}_1\ \hat{x}_2]^T$ and using the assumption N large ($N \to \infty$), we obtain that the autocorrelation matrix for $[\hat{x}_1\ \hat{x}_2]^T$ is K and the autocorrelation matrix for $[i_1\ i_2]^T$ is $K_i = K/\Delta^2$.

2.2. Integer-to-integer transform

We have to define an integer-to-integer transform: $[i_1\ i_2] \mapsto [i_1'\ i_2']$ which depends on a set of parameters. A first tempting choice is to approximate a linear transform $T = [T_{11}\ T_{12}; T_{21}\ T_{22}]$ (the entries of matrices are described following Matlab–like notations). If $T_{12} \ne 0$ the matrix T can be factored by using the lifting scheme as follows

$$T = \begin{bmatrix} 1 & 0 \\ q_3 & 1 \end{bmatrix}\begin{bmatrix} 1 & q_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ q_1 & 1 \end{bmatrix} \tag{6}$$

In [1] the transformed vector is expressed in the real-valued coordinates $[i_1\Delta\ i_2\Delta]$. A first matrix multiplication $T_1 = [1\ 0; q_1 1]$ creates $[i_1\Delta,\ q_1i_1\Delta + i_2\Delta]$ which is truncated to $[i_1\Delta,\ (\lfloor q_1i_1 + 0.5\rfloor + i_2)\Delta]$. It is simpler to define this stage in integer coordinates as $[i_1,\ i_2] \mapsto [i_1,\ \lfloor q_1i_1 + 0.5\rfloor + i_2]$. Continue the second multiplication, by $T_2 = [1\ q_2; 0\ 1]$ producing after rounding $[i_1 + \lfloor q_2\lfloor q_1i_1 + 0.5\rfloor + q_2i_2 + 0.5\rfloor,\ \lfloor q_1i_1 + 0.5\rfloor + i_2]$. Finally, the third multiplication, by $T_3 = [1\ 0; q_3\ 1]$, produces after rounding $[i_1 + \lfloor q_2\lfloor q_1i_1 + 0.5\rfloor + q_2i_2 + 0.5\rfloor,\ \lfloor q_3i_1 + q_3\lfloor q_2\lfloor q_1i_1 + 0.5\rfloor + q_2i_2 + 0.5\rfloor + 0.5\rfloor + \lfloor q_1i_1 + 0.5\rfloor + i_2]$. Since there is an one-to-one mapping between the vectors $[x_1\ x_2]$ and $[i_1\ i_2]$ we denote by $\hat{T}(\hat{x})$ the mapping $[i_1\ i_2] \mapsto [i_1'\ i_2']$. There are three rounding operations, introducing the errors ε_1, ε_2 and ε_3 respectively. For $\hat{x} = [i_1\Delta\ i_2\Delta]^T$, it may be proved[1] that $||\hat{T}(\hat{x}) - T\hat{x}||_\infty$ is less or equal than

$$\left(1 + max\left\{|T_{12}|, |T_{22}| + |T_{12}^{-1}(T_{22}-1)|\right\}\right)\frac{\Delta}{2} \tag{7}$$

This shows that $\hat{T}(\hat{x})$ is a good approximation for T (in the sense of the norm $||\cdot||_\infty$) when Δ is small.

Since the transformation $[i_1\ i_2] \mapsto [i_1'\ i_2']$ is completely reversible[1], the pmf of $[i_1'\ i_2']$ is exactly the same as that of $[i_1\ i_2]$. An equally likely contour in the $\mathcal{Z}^2$ lattice (which, according to (5) is the ellipsis $[i_1\ i_2]K^{-1}[i_1\ i_2]^T = Const$) will be transformed to the approximate ellipsis $[i_1'\ i_2']T^{-T}R^{-1}T^{-1}[i_1'\ i_2']^T = Const$ in the new coordinate axes.

2.3. The coding rate versus the algorithmic complexity

Using the same quantizer $Q_{N,\Delta}$ for all the coding schemes (DQ, UTC, I_2ITC) ensures nearly the same distortion in the high resolution hypothesis. So, the performance of the encoding algorithm is given by the trade-off between rate and complexity. A comparison of coding rates for different methods can be obtained by studying the entropy of the encoded indexes.

For the UTC scheme we denote $\underline{y} = U\underline{x}$ and $\hat{\underline{y}}$ is the quantized version for $\underline{y}$. In the I_2ITC case $\hat{\underline{z}}$ is the notation for $\hat{T}(\hat{\underline{x}})$. $R_s(\cdot)$ is the rate in bits/sample for scalar entropy coding (based on the marginal probabilities of the vector entries) and $R_v(\cdot)$ is the rate for vector entropy coding (evaluated with joint probabilities of the vector entries). The rate $R_v \triangleq R_v(\hat{\underline{z}}) = R_v(\hat{\underline{x}}) \approx R_v(\hat{\underline{y}})$ obtained by entropic coding based on joint probabilities is the same rate for all three schemes. Unfortunately, to achieve this rate, the size of the code dictionary (necessary to implement Huffman coding) is N^2. The scalar coding of the vector entries reduces the memory requirement to at most $2N$ codewords, and we evaluate next the price to be payed for this reduction. Generally $R_s(\underline{\hat{x}}) \geq R_v$, but $R_s(\underline{\hat{y}}) = R_v$ when $\underline{x}$ is jointly Gaussian, making UTC attractive since it gives similar results to vector coding with a smaller amount of resources. It is proved in [1] that relaxing UTC to non-orthogonal transform, the best result with I$_2$ITC is $R_s(\underline{\hat{z}}) = R_s(\underline{\hat{y}})$. The same rate was obtained[1] by using a single (common) codebook for both channels, which reduces the memory requirements from $2N$ codewords to N codewords. To achieve this performance, the transform T was chosen such that $T^{-T}K^{-1}T^{-1}$ is proportional to the identity matrix I, transforming equally likely contours from ellipsis to circles. Two assumptions are needed for the above to hold: $\hat{T} \approx T$, and $N = 2^{M+1} - 1$ is a large number.

Our approach is a further step towards reducing the complexity: we apply GR codes and, consequently, eliminate the need to store Huffman tables. GR codes will be applied independently for each channel, the average code length being computed as $L_{av} = (L_{av}^1 + L_{av}^2)/2$ where L_{av}^i denotes the average code length for channel i. The problem to be solved is to design the mapping $\hat{T}$ that ensures the minimum L_{av} when $\underline{x}$ is jointly Gaussian with known pdf and the quantizer $Q_{N,\Delta}$ is given.

2.4. An approach based on Karhunen-Loeve transform

We choose $T = U$ orthogonal, where $UKU^T = diag(s_1^2, s_2^2)$. From the transform U we obtain the integer to integer mapping $\hat{U}(\cdot)$ by performing the lifting factorization described in (6). The equality $[\hat{x}_1' \ \hat{x}_2']^T = U[\hat{x}_1 \ \hat{x}_2]^T$ implies

$$P([\hat{x}_1' \ \hat{x}_2']^T) \approx Ce^{-\frac{1}{2}\left(\frac{\hat{x}_1'}{s_1}\right)^2} e^{-\frac{1}{2}\left(\frac{\hat{x}_2'}{s_2}\right)^2}$$

where we use the approximations (4) and (7). The result shows that the random variables $\hat{x}_1'$ and $\hat{x}_2'$ are independent and we can minimize the average code length by encoding separately each variable. Since $|U| = 1$ we find that $s_1^2 s_2^2 = (1-\rho^2)\sigma_1^2\sigma_2^2$. The equation above gives the pmf for i_1' as $P(i_1') = C_1 e^{-\frac{1}{2}\left(\frac{i_1'}{s_1'}\right)^2}$ where $s_1' = s_1/\Delta$ and respectively the pmf for i_2' is $P(i_2') = C_2 e^{-\frac{1}{2}\left(\frac{i_2'}{s_2'}\right)^2}$, $s_2' = s_2/\Delta$. We consider now a supplementary integer-to-integer reversible transform $[i_1' \ i_2'] \to [i_1'' \ i_2'']$. When GR_{k_1} is applied for channel 1 and GR_{k_2} for channel 2, the average code length is given by $L_{av} = \frac{1}{2}(L_{av}^1 + L_{av}^2) =$
$\frac{1}{2}\sum_{p=1}^{2}\left\{2\sum_{i_p''=0}^{\infty}\left\lfloor\frac{i_p''}{2^{k_p}}\right\rfloor C_p'' \exp\left(-\frac{1}{2}\left(\frac{i_p''}{s_p''}\right)^2\right) + (k_p+1)\right\}$.

A fast method to select the parameters k_p, $p \in \{1,2\}$ is to choose

$$2^{k_p} = \lfloor f s_p'' \rfloor, \ \ f \in (0,1) \tag{8}$$

where $f = 0.6745$, as explained in Section 3. The term $k_1 + k_2$ in L_{av} can be approximated by $\log_2(fs_1'') + \log_2(fs_2'') = \log_2(s_1''s_2''f^2)$ and it is invariant with respect to any unitary determinant transform $[i_1' \ i_2'] \to [i_1'' \ i_2'']$ since $s_1''s_2'' = s_1's_2'$.

We focus on the transform $[i_1' \ i_2'] \to [i_1'' \ i_2'']$ which equalizes the variances $s_1'' = s_2'' = s''$ [1] in order to accomplish the MD requirements. The pmf for i_1'' is the same as the pmf for i_2'', $P(i'') = C(s'')e^{-\frac{1}{2}\left(\frac{i''}{s''}\right)^2}$. We will evaluate in the sequel the average code length L_{av} after this transform. When GR_k code is used for both channels, the term of L_{av} which depends on the transform, is

$$\Lambda(s'') = 2\sum_{i''=0}^{\infty}\left\lfloor\frac{i''}{2^k}\right\rfloor C(s'')\exp\left(-\frac{1}{2}\left(\frac{i''}{s''}\right)^2\right).$$

Lemma 2.1 *If $s'' > 1/\sqrt{2\pi}$, then the normalization constant $C(s'')$ is bounded:*

$$\frac{1}{\sqrt{2\pi}s''+1} \leq C(s'') \leq \frac{1}{\sqrt{2\pi}s''-1}$$

Proof: We apply the Maclaurin-Cauchy integral test to the series $\sum_{i''=0}^{\infty} e^{-\frac{1}{2}\left(\frac{i''}{s''}\right)^2} = \sum_{i''=0}^{\infty} f(i'')$ where $f : [0,\infty) \to (0,1]$, $f(x) = e^{-\frac{1}{2}\left(\frac{x}{s''}\right)^2}$ is continuous and monotone decreasing. The nth partial sum of the considered series is $a_n = \sum_{i''=0}^{n} e^{-\frac{1}{2}\left(\frac{i''}{s''}\right)^2}$. Consider $b_n = \sum_{i''=0}^{n}\int_{i''}^{i''+1} f(x)dx = \int_0^{n+1} f(x)dx$. For any integer $N_1 > 0$, $a_{N_1+1} - 1 \leq b_{N_1} \leq a_{N_1}$ which implies that the series $(a_n)_{n\geq 0}$ converges to L_a and $\frac{\sqrt{2\pi}s''}{2} \leq L_a \leq \frac{\sqrt{2\pi}s''}{2} + 1$ (we used $lim_{N_1\to\infty} b_{N_1} = \frac{\sqrt{2\pi}s''}{2}$). $\square$

Lemma 2.2 *a) For any positive integer i'',*

$$\frac{1}{\sqrt{2\pi}s''+1}\frac{i''-fs''}{fs''} \leq C(s'')\left\lfloor\frac{i''}{2^k}\right\rfloor \leq \frac{1}{\sqrt{2\pi}s''-1}\frac{i''}{fs''-1}$$

b) Using the approximation $\frac{1}{\sqrt{2\pi}s''-1}\frac{i''}{fs''-1} \approx \frac{1}{f\sqrt{2\pi}}\frac{i''}{s''^2}$ we obtain

$$\sum_{i''=0}^{\infty}\left\lfloor\frac{i''}{2^k}\right\rfloor C(s'')\exp\left(-\frac{1}{2}\left(\frac{i''}{s''}\right)^2\right) \leq$$
$$\frac{1}{f\sqrt{2\pi}}\sum_{i''=0}^{\infty}\frac{i''}{s''^2}\exp\left(-\frac{1}{2}\left(\frac{i''}{s''}\right)^2\right)$$

Proof: a) The condition (8) leads to $\frac{1}{fs''-1} > \frac{1}{2^k} \geq \frac{1}{fs''}$ if $fs'' - 1 > 0$. By applying Lemma 2.1 we obtain the result. $\square$

Proposition 2.1 *Under the assumptions from Lemma 2.1 and 2.2, the variable term in L_{av} is bounded by*

$$\Lambda(s'') \leq \frac{2}{f\sqrt{2\pi}} + \frac{2}{f\sqrt{2\pi e}}\frac{\Delta}{\sqrt{s_1 s_2}} \tag{10}$$

Proof: Let consider the function $\phi : [0,\infty) \to [0,\infty)$, $\phi(x) = \frac{x}{s''^2}\exp\left(-\frac{1}{2}\left(\frac{x}{s''}\right)^2\right)$ which has the derivative $\phi'(x) = \frac{1}{s''^2}\exp\left(-\frac{1}{2}\left(\frac{x}{s''}\right)^2\right)\left[1-\left(\frac{x}{s''}\right)^2\right]$. It is easy to observe that the function $\phi(\cdot)$ is increasing on the interval $[0, s'']$ and decreasing on the interval (s'', ∞). The monotonicity implies for any $i'' \in \{0, 1, \ldots, \lfloor s''\rfloor - 1\}$, $\phi(i'') \leq \int_{i''}^{i''+1}\phi(x)dx \leq \phi(i''+1)$ and for any integer $i'' \geq \lfloor s''\rfloor + 1$, $\phi(i''+1) \leq \int_{i''}^{i''+1}\phi(x)dx \leq \phi(i'')$. With the assumption that the supplementary inequality $\phi(\lfloor s''\rfloor + 1) \leq \int_{\lfloor s''\rfloor}^{\lfloor s''\rfloor+1}\phi(x)dx \leq \phi(\lfloor s''\rfloor)$ is true, we obtain $\sum_{i''=0}^{\infty}\frac{i''}{s''^2}\exp\left(-\frac{1}{2}\left(\frac{i''}{s''}\right)^2\right) \leq \int_0^{\infty}\phi(x)dx + \phi(\lfloor s''\rfloor)$. This inequality, Lemma 2.2b) and the property $s'' = \frac{\sqrt{s_1 s_2}}{\Delta}$ imply (10). $\square$

Since the upper bound in (10) is approximately 1.2 when Δ is small, we can conclude that the average lengths, with and without the transform $[i_1' \; i_2'] \to [i_1'' \; i_2'']$, differ by at most 1.2 bits. Note that the parameter k was computed to fulfill (8). We will investigate in the next Section how far from optimal is the code parameter k obeying the condition (8).

3. SELECTION OF CODE PARAMETER WHEN THE INPUT DISTRIBUTION IS GAUSSIAN

Consider a Gaussian r.v. x and denote $g(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right]$. The r.v. x is quantized to $\hat{x} = Q_{N,1}(x)$ as in (2), which leads to the pmf for the modulus $\tilde{x} = |\hat{x}|$:

$$Prob\{\tilde{x} = n\} =$$

$$\begin{cases} \int_{-1/2}^{1/2} g(x)dx, \; n = 0 \\ 2\int_{n-1/2}^{n+1/2} g(x)dx, \; 0 < n < 2^M - 1 \\ 2\left\{\int_{2^M-3/2}^{2^M-1/2} g(x)dx + \int_{2^M-1/2}^{\infty} g(x)dx\right\}, \; n = 2^M - 1 \end{cases}$$

Proposition 3.1 *The average code length obtained when $\tilde{x}$ is encoded with GR_k, $k = M - m$, is given by*

$$EL_k^{\tilde{x}} = EL_{M-m}^{\tilde{x}} =$$

$$\begin{cases} M + 1, & m = 0 \\ M + 2^m - m - 2\sum_{p=1}^{2^m-1} erf\left(p\frac{r}{2^m} - \delta\right), & 1 \le m \le M \end{cases}$$

where $r = \frac{2^M}{\sigma}$, $erf(t) = \frac{1}{\sqrt{2\pi}}\int_0^t \exp(-\frac{1}{2}y^2)dy$ and $\delta = \frac{r}{2^{M+1}}$.

Corollary 3.1 *For any integer m, $2 \le m < M$, if $EL_{M-m}^{\tilde{x}} > M$ then $EL_{M-(m+1)}^{\tilde{x}} > EL_{M-m}^{\tilde{x}}$.*

The problem of finding the parameter k (or equivalently m) such that the code GR_k insures the shortest average code length for the moduli $\tilde{x}$ can be solved by applying a full-search algorithm which sequentially evaluates $EL_{M-m}^{\tilde{x}}$ for all possible values of m: $0, 1, \ldots, M$. The result from Corollary 3.1 reduces the complexity of such an algorithm: the different values of m are tested in increasing order and the process is stopped when the first $m^* \ge 2$ with property $EL_{M-m^*}^{\tilde{x}} > M$ is found. It results from Proposition 3.1 that when we use GR_M ($m = 0$) to code the samples of $\tilde{x}$ truncated to the values $\{0, 1, \ldots, 2^M - 1\}$, the average code length $(M + 1)$ is 1 bit longer than the code length necessary to represent the samples in binary format. It is straightforward to check that $M < EL_k^{\tilde{x}} < M + 1$ if $k = M - 1$ ($m = 1$).

The condition (1) for selecting the parameter k becomes

$$k = \lfloor \log_2(0.6745\sigma) \rfloor \Leftrightarrow k = \lfloor \log_2(\sigma) - 0.57 \rfloor . \qquad (11)$$

A comparison of results obtained by applying the minimum average code length criterion and the condition above is represented in Figure 1. We observe the tendency of criterion introduced in [4] to commute the code parameter from k to $k-1$ for values of standard deviation σ which are larger than in the case of applying minimum average code length criterion. This leads to longer average code for some values of the standard deviation, but the difference is not dramatic. An improvement in average code length can be obtained

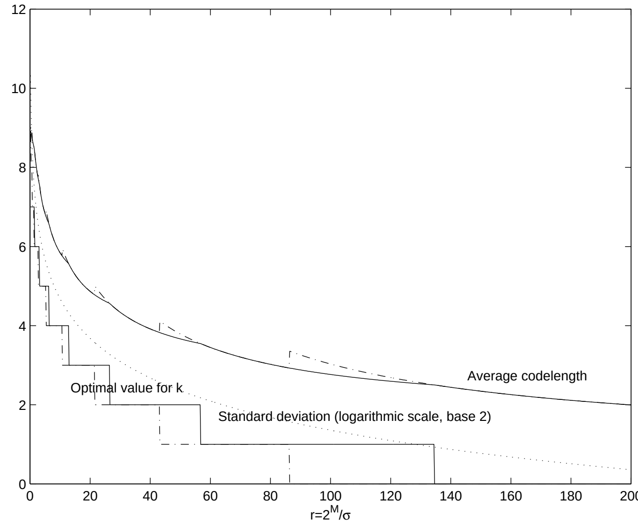

Fig. 1. Comparison of average code length [bits/sample] in the case of minimum average code length criterion (continuous line) and criterion from [4] (dashed line) for different values of index $r = 2^M/\sigma$ where $M = 8$. For each case the optimal value of the parameter k is plotted by using the same type of line as for the average code length. The standard deviation σ is represented in logarithmic scale for comparison with commutation values in selection of k.

by defining thresholds (as a function of the variance) for switching between the selection of k or $k - 1$ as coding parameter. In this case it is necessary to store a table containing the thresholds. The same table may be shared by all channels of the integer-to-integer transform.

4. REFERENCES

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