ASYMPTOTIC CAPACITY OF SPACE-TIME CODING FOR ARBITRARY FADING: A CLOSED FORM EXPRESSION USING GIRKO'S LAW

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ABSTRACT

Several works addressed the problem of deriving the asymptotic capacity of a wireless system with space diversity in random fading. However, the theory of random matrices was never used in evaluating the asymptotic optimal performance in closed form. By increasing the number of transmit and receive antennas the resulting capacity tend to be a stable value independent of the fading realization. This surprising result is a consequence of Girko's law, stating that the asymptotic distribution of the eigenvalues of a random matrix, with independent identically distributed zero mean complex entries, is a circle. The conditions on the probability density function of the matrix entries are satisfied by the majority of random non-line of sight fading models. Using this theory in this paper we derive the close form expression for the asymptotic capacity of a system with transmit and receive diversity, assuming independent flat fading for each transmit-receive antenna link, with equal distribution. Our formula fits the numerical results even if the number of transmit an receive antennas as small as ten.

1. INTRODUCTION AND PROBLEM STATEMENT

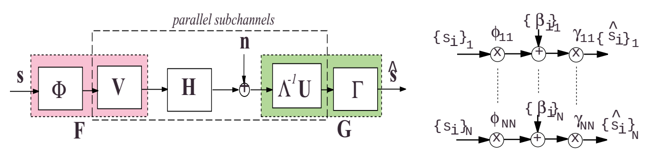
The progressive saturation of the radio frequency resources and the increasing demand for high-bit rate wireless communications generated an increasing interest in the use of multiple transmit and receive antennas to augment the channel capacity. The optimization of the information rate through multiple channels was considered since the early work on the optimal coding for multi-input multi-output (MIMO) models in [1]. More recently, in the context of space time coding for flat fading in [5], [3], and for frequency selective channels in [7], it was shown that the capacity can increase linearly with the minimum between the number of transmitters and the number of receivers. With $\mathbf{H}$ denoting the array manifold, the input output relationship is:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{1}$$

where we assume that $\mathbf{n}$ is an additive white Gaussian noise (AWGN) vector $\sim \mathcal{N}(\mathbf{0}, \sigma_n^2 \mathbf{I})$. Assuming that CSI is available at both the transmitter and receiver sides under a constraint on the transmit power, a number of different optimization criteria can be solved assuming that a linear precoder $\mathbf{F}$ and decoder $\mathbf{G}$ are used at the transmitter and the receiver [6, Ch.9], so that

$$\hat{\mathbf{s}} := \mathbf{G}\mathbf{y} = \mathbf{G}\mathbf{H}\mathbf{F}\mathbf{s} + \mathbf{G}\mathbf{n}. \tag{2}$$

Invariably, the best linear mapping consist in modulating the symbols using the eigenvectors of the channel as illustrated in Fig.1 (a) where $\mathbf{H} = \mathbf{U}^* \Lambda \mathbf{V}$, with * indicating transpose and conjugate, is the singular value decomposition (SVD) of $\mathbf{H}$ and Φ, Γ are diagonal matrices.



(a) Matrix form (b) AGN subchannels

Fig. 1. Optimum precoding /decoding.

Fig.1 (b), where $\beta_i \sim \mathcal{N}(0, \sigma_n^2/\lambda_i^2)$ are uncorrelated AGN random variables, shows how the channel is equivalently decomposed by $\mathbf{F}$ and $\mathbf{G}$ in a set of parallel independent Gaussian subchannels where, in case of white additive Gaussian noise, the signal to noise ratio is [6, Ch.9]

$$SNR_i = \phi_{ii}^2 \lambda_i^2 / \sigma_n^2, \tag{3}$$

and ϕ_{ii}^2 controls the power allocated on the ith equivalent subchannel. The obvious question that arises is how much can be asymptotically gained by increasing the number of subchannels, which is obtained by increasing the minimum between the number of receive and tranmit antennas. Our approach here differs from the one in [5], [3] because, we explicitly derive the capacity for any realization of a very large $\mathbf{H}$ exploiting one of the most interesting phenomena arising in the theory of random variables after the central limit theorem: the so called semi-circle or circle law, the last named after Girko [4], that generalized earlier results by Wiener to complex matrices.

Next section specifies how the channel capacity can be derived as a function of the singular values of $\mathbf{H}$ while in the Section 3 we will explain how Girko's law can be used to predict the asymptotic gain obtainable by increasing the number of receive and transmit antennas under wide conditions on the fading distribution. For simplicity we will assume that:
(a1) The number of receive and transmit antennas coincide ($\mathbf{H}$ is square and of size N).
(a2) The elements of $\mathbf{H}$ $H_{i,j}$ are independent identically distributed (i.i.d.) random variables with $E\{H_{i,j}\} = 0$ and $Var[H_{i,j}] = \sigma_H^2$.

2. PRELIMINARIES

If the average transmit power is limited

$$trace(\mathbf{F}\mathbf{F}^*) = \sum_{n=1}^{N} \phi_{nn}^2 = \mathcal{P}_0. \tag{4}$$

The p.d.f of the input vector $\mathbf{x}$ that maximizes the channel mutual information $I(\mathbf{y}, \mathbf{x})$ under (4) is $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_{xx})$:

$$I(\hat{\mathbf{s}}, \mathbf{s}) = \log |\mathbf{I} + \mathbf{H}\mathbf{R}_{xx}\mathbf{H}^*\sigma_n^{-2}| \tag{5}$$

and (5) is maximized when $\mathbf{R}_{xx}$ and $\mathbf{H}^*\mathbf{H}$ have the same eigenvectors [2]. This is exactly what is obtained with $\mathbf{x} = \mathbf{F}\mathbf{s}$ with $\mathbf{s} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$[1] and $\mathbf{F} = \mathbf{\Phi}\mathbf{V}$ [6, Ch.9], hence $\max_x I(\hat{\mathbf{s}}, \mathbf{s}) = \sum_{n=1}^{N} \log(1 + SNR_n)$ and the asymptotic capacity

$$C = \lim_{N\to\infty} \frac{1}{N} \max_{\phi_{11},\ldots,\phi_{NN}} \sum_{n=1}^{N} \log(1 + SNR_n). \tag{6}$$

The maximization with respect ϕ_{nn} under the average power constraint in (4), leads to the so called *water-filling*:

$$\phi_{kk}^2 = \left(K - \sigma_n^2/\lambda_{kk}^2\right)^+ \tag{7}$$

where $K = const.$ is determined by imposing (4) and $^+$ indicates that when the function inside the parenthesis is negative the value has to be set to zero. Assuming that λ_{kk}^2 are sorted in decreasing order and using (7), for every $\mathcal{P}_0 > 0$ we can find a maximum $\bar{N}\ :\ 1 \leq \bar{N} \leq N$ such that for $k \leq \bar{N} \Rightarrow \phi_{kk}^2 > 0$ and for $k > \bar{N} \Rightarrow \phi_{kk}^2 = 0$ ($\bar{N} = 1$ is the trivial case where $\phi_{11}^2 = \mathcal{P}_0 > 0$). Replacing (7) in (3) we can write (6) simply as

$$C = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{\bar{N}} \log(K\lambda_{kk}^2/\sigma_n^2). \tag{8}$$

[1] We assume that the symbols are white, but we can always include a symbol 'whitener' in $\mathbf{F}$.

Oftentimes, when $\mathcal{P}_0 \gg 1$ the formula is approximated using uniform loading, i.e. $\phi_{kk}^2 \approx \mathcal{P}_0/N$ in which case

$$C = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \log\left(1 + \frac{\mathcal{P}_0\lambda_{kk}^2}{N\sigma_n^2}\right) \tag{9}$$

For finite N the dependence of the eigenvalues distribution on the distribution of $H_{i,j}$ is rather intricate. However, similar to what happens with the sum of random variables, the distribution becomes invariant with respect to the particular density of the entries $H_{i,j}$ as $N \to \infty$, as explained in the following section.

3. SPACE-TIME CODING AND GIRKO'S LAW

The usefulness of Girkos's law in the analysis of space-time precoding lies in equation (1) and in Fig.1 (b). The last clearly shows that the performance of a system described as in (1) are function of nothing but the eigenvalues of $\mathbf{H}$. Since $\mathbf{H}$ is random, if two realization of $\mathbf{H}$ have the same eigenvalues then the performance in additive Gaussian noise are identical. The surprising implication of the theory of random matrices is that, under the assumptions specified at the end of Section 1, according to Girko's law as $N \to \infty$ the eigenvalues of $\mathbf{H}/\sqrt{\sigma_H^2 N}$ tend to fill a circle of radius one, as illustrated in Fig.3. From distribution of the the

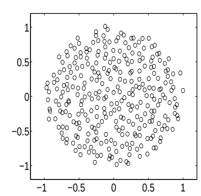


Fig. 2. $\lambda_n\sqrt{\sigma_H^2 N}$ of a complex Gaussian $\mathbf{H}$ for $N = 300$.

eigenvalues we can obtain the distribution of the singular values, since the last are the absolute values of the eigenvalues. Thus singular values $\tilde{\lambda}_n$ of the normalized matrix $\tilde{\mathbf{H}} = \mathbf{H}/\sqrt{\sigma_H^2 N}$ as $N \to \infty$ tend to assume the value x with probability density function (p.d.f) the is a fourth of a circle:

$$p_{\tilde{\lambda}}(x) = 2/\pi\sqrt{1 - x^2/4} \qquad 0 \leq x \leq 2. \tag{10}$$

Further details on Girko's law can be found in [4]. Using (10) we can write

$$\lim_{N\to\infty} \frac{SNR_n}{N} = SNR(x) = \gamma\tilde{\phi}^2(x)x^2, \tag{11}$$

where γ is defined as:

$$\gamma = \sigma_H^2/\sigma_n^2 \tag{12}$$

4. ASYMPTOTIC CAPACITY

For $N \to \infty$ (11) and (10):

4.1. Numerical results

(a) Uniform loading (b) Optimum Loading

Fig. 3. Comparison between theory and simulation.

5. REFERENCES

[1] L. H. Brandenburg and A. D. Wyner, "Capacity of the Gaussian channel with memory: The multivariate case," *Bell Syst. Tech. Journ.*, Vol. 53, pp. 745-778, May/June 1974.

[2] T. M. Cover, J. A. Thomas, *Elements of Information Theory*, New York: John Wiley & Sons, Inc., 1991.

[3] G.J. Foschini, "Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas," *Bell Labs. Tech. Journ.*, Vol. 1, No. 2, pp. 41–59, 1996.

[4] V.L. Girko, "On the circle law", *Theor. Probability and Math. Statist.*, pp. 15–23, 1984.

[5] T.L. Marzetta, B. H. Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading", *IEEE Trans. on Info. Theory*, Vol. 45, No. 1, pp.139-157, Jan. 1999.

[6] "Signal Processing Advances in Communications," G. B. Giannakis, L. Tong, P. Stoica, Y. Hua, (editors), Prentice Hall, 2000.

[7] G. G. Raleigh and J. M. Cioffi, "Spatiotemporal coding for wireless communications," *IEEE Trans. on Commun.*, Vol. 46, No. 3, pp. 357–366, March 1998.