QR BASED ITERATIVE UNBIASED EQUATION ERROR FILTERING

Bruce E. Dunne[1] and Geoffrey A. Williamson[2]

[1]Tellabs Research Center, 3740 Edison Lakes Pkwy., Mishawaka, IN 46545 USA
[2]Dept. of Elec. and Comp. Engr., Illinois Institute of Technology, Chicago, IL 60616 USA

ABSTRACT

A QR-decomposition based algorithm is presented for unbiased, equation error adaptive IIR filtering. The algorithm is based on casting the adaptive IIR filtering in a mixed Least Squares - Total Least Squares (LS-TLS) framework. This formulation is shown to be equivalent to the minimization of the mean-square equation error subject to a unit norm constraint on the denominator parameter vector. An efficient implementation of the mixed LS-TLS solution is achieved through the use of back substitution and inverse iteration. Unbiasedness of the system parameter estimates is established for the mixed LS-TLS solution in the case of uncorrelated output noise, and the algorithm is shown to converge to this solution.

1. INTRODUCTION

Equation error adaptive IIR filters are known to suffer from parameter bias when the desired filter output is measured in the presence of noise, even when that noise is uncorrelated [5]. A number of investigators have addressed this issue by recasting the problem as one of minimizing the mean-square equation error subject to a unit norm constraint on the denominator polynomial's coefficient vector, rather than the traditional monic constraint, for example [1] and [6]. Doing so reinterprets the equation error IIR problem as a Total Least Squares (TLS) minimization rather than one of Least Squares (LS), whether or not this is explicitly noted. It is more correct, however, to label this problem as mixed LS-TLS. The TLS formulation assumes noise on all the data while in mixed LS-TLS some data is noisy and other data is not [4]. As the filter inputs remain noiseless, the equation error IIR adaptive filter problem is within the mixed LS-TLS class.

In this paper we develop a mixed LS-TLS based algorithm to minimize the mean-squared equation error. The algorithm is based on the QR-decomposition approach to mixed LS-TLS described in [2]. The first stage of our iterative algorithm addresses a reduced dimension TLS problem and employs inverse iteration, efficiently implemented via back substitution, as has been done in the full TLS setting ([7], [8]). The second stage concerns an embedded LS problem, which is also efficiently implemented via back substitution, exploiting the QR decomposition.

By connecting our work to that of [6], we demonstrate that the algorithm asymptotically produces an unbiased estimate of the system parameters. We provide simulations corroborating this algorithm behavior.

2. SYSTEM DESCRIPTION

Consider the system identification problem, as shown in Figure 1. We wish to model the unknown causal stable system h(k) by observing the system input and outputs. The input u(k) drives the system, yielding v(k), which is then corrupted by the uncorrelated noise sequence n(k).

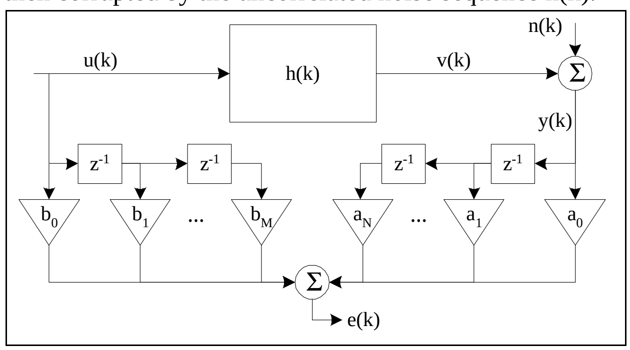

Figure 1: Equation Error Filtering Diagram

We can write the system relationship as:

$$y(k) = h(k)*u(k) + n(k) \tag{1}$$

where h(k) is modeled as an IIR filter such that

$$H(z) = \frac{B(z)}{1-A(z)} \tag{2}$$

We will assume that B(z) is of order M and A(z) is of order N. In (2), we imply that A(z) has been scaled such that $a_0 = -1$, although this normalization is a convenience in order to match the input/output relationship.

We define the regressor vectors

$$\begin{aligned} U(k) &= \left[u(k)\ u(k-1)\ \dots\ u(k-M)\right]^T \\ Y(k) &= \left[y(k)\ y(k-1)\ \dots\ y(k-N)\right]^T \\ \bar{Y}(k) &= \left[y(k-1)\ y(k-2)\ \dots\ y(k-N)\right]^T \end{aligned} \tag{3}$$

and composite regressor vector $\phi(k) = \left[U(k)\ Y(k)\right]^T$.

Let γ be defined as the extended filter parameter weights, where

$$\gamma = \left[\gamma_1\ \gamma_2\right]^T \qquad \gamma_1 = \theta_1 \qquad \gamma_2 = \left[a_0\ \theta_2\right]^T$$
$$\theta_1 = \left[b_0\ b_1\ \dots\ b_M\right]^T \qquad \theta_2 = \left[a_1\ a_2\ \dots\ a_N\right]^T \tag{4}$$

It is often convenient to normalize γ to have $a_0 = -1$ such that the usual input/output relationship is satisfied. In this case, we define

$$\hat{\gamma} = \left[\hat{\theta}_1^T\ -1\ \hat{\theta}_2^T\right]^T \tag{5}$$

3. EQUATION ERROR APPROACHES

3.1. Mixed LS-TLS

In the mixed LS-TLS problem, the goal is to find a solution to a system of equations where a portion of the input data and the output vector are noisy and the remaining input data are known exactly.

Suppose that we are trying to solve a system of equations $A\theta = \kappa$. Furthermore, assume that A can be segmented into its noise-free columns (A_1) and its noisy columns (A_2). We can then write

$$\left[A_1\ A_2\right]\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} \approx \kappa \tag{6}$$

The goal of the mixed LS-TLS solution is to find $\bar{A}_2$ and $\bar{\kappa}$ such that the pertubation to the original system is minimized in a Frobenius sense, i.e.,

$$\min\left\|\left[A_2\ \kappa\right] - \left[\bar{A}_2\ \bar{\kappa}\right]\right\|_F \tag{7}$$

subject to the constraint that

$$\bar{\kappa} \subseteq \Re\left(\left[A_1\ \bar{A}_2\right]\right) \tag{8}$$

Given $\bar{A}_2$ and $\bar{\kappa}$, the LS method is used to find θ as the minimum norm solution to

$$\left[A_1\ \bar{A}_2\right]\theta = \bar{\kappa} \tag{9}$$

The solution to the mixed LS-TLS problem is given in [2]. A partial QR factorization of the matrix $C = [A_1\ \kappa\ A_2]$ is performed to give

$$Q_{part}^T C = Q_{part}^T\left[A_1\ \kappa\ A_2\right] = \begin{bmatrix}R_{11} & R_{1\kappa} & R_{12}\\ 0 & R_{2\kappa} & R_{22}\end{bmatrix} \tag{10}$$

such that R_{11} is an upper triangular square matrix of dimension equal to the number of noise-free columns. First, the solution for the reduced dimension TLS problem is found for (see [4] for a discussion of determining the TLS solution)

$$R_{22}\theta_2 \approx R_{2\kappa} \tag{11}$$

Given $\theta_2{}^*$, θ_1 can be determined by solving the LS system of equations

$$R_{11}\theta_1 \approx R_{1\kappa} - R_{12}\hat{\theta}_2{}^* \tag{12}$$

The equation error iterative problem can easily be cast into the mixed LS-TLS framework. At each k-th iteration, the data matrices are augmented by a new row. A_1 is augmented by U(k), A_2 by $\bar{Y}(k)$ and κ with y(k). We formulate the iterative equation error technique in a similar manner but in such a way that the dimension of the problem does not grow with each new data point.

3.2. Equation Error Filtering

In the statistical equation error approach, the goal is to minimize the mean squared error $E\{e^2(k)\}$ subject to a unit norm constraint on γ_2. This error can be expressed as $e(k) = \phi^T(k)\gamma$. Assuming that the noise is uncorrelated with the signal, the mean squared error is:

$$E\{e^2(k)\} = \left[\gamma_1^T\ \gamma_2^T\right]\begin{bmatrix}R_{uu} & R_{uv}^T\\ R_{uv} & R_{yy}\end{bmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix} \tag{13}$$

The solution (see [6]) is a two step process. First, γ_2 is set to the eigenvector associated with the smallest eigenvalue of the matrix $R_{y/u}$, where $R_{y/u}$ is:

$$R_{y/u} = R_{yy} - R_{uv}^T R_{uu}^{-1} R_{uv} \tag{14}$$

The normalized $\hat{\gamma}_2{}^*$ is then used to compute γ_1:

$$\gamma_1 = -R_{uu}^{-1} R_{uv}\hat{\gamma}_2{}^* \tag{15}$$

3.3. Equivalence of the Two Solutions

We now show that the mixed LS-TLS solution is equivalent to the statistical equation error solution.

To start, we recognize that the sample autocorrelation matrix approaches the true correlation matrix as the data record gets long:

$$\frac{1}{k}C^T(k)C(k) = E\{\phi(k)\phi^T(k)\} \text{ as } k \to \infty \tag{16}$$

Assuming that the iteration number is large, we can ignore the dependence on k without loss of generality. Recall that normalization of $a_0 = -1$ is necessary to be consistent with the input/output relationship, hence the scaling due to k is superfluous.

In (10), the QR factorization is composed of unitary transforms, hence it is true that

$$\begin{bmatrix} R_{11} & R_{1\kappa} & R_{12} \\ 0 & R_{2\kappa} & R_{22} \end{bmatrix}^T \begin{bmatrix} R_{11} & R_{1\kappa} & R_{12} \\ 0 & R_{2\kappa} & R_{22} \end{bmatrix} = \begin{bmatrix} R_{uu} & R_{uv}^T \\ R_{uv} & R_{yy} \end{bmatrix} \quad (17)$$

Expanding (17), equating like size terms and rearranging, it is easy to show that

$$\begin{bmatrix} R_{2\kappa} & R_{22} \end{bmatrix}^T \begin{bmatrix} R_{2\kappa} & R_{22} \end{bmatrix} = R_{y/u} \quad (18)$$

In section 3.1, $\theta_2{}^*$ is the last N components of the right singular vector associated with the smallest singular value of $[R_{2\kappa}\ R_{22}]$. In section 3.2, $\gamma_2{}^*$ is the eigenvector associated with the smallest eigenvalue of $R_{y/u}$. Given the equivalence of (18), by inspection it must be true that

$$\begin{bmatrix} -1 & \hat{\theta}_2{}^* \end{bmatrix}^T = \hat{\gamma}_2{}^* \quad (19)$$

To see the equivalence of θ_1 and γ_1, we can rewrite (12) as

$$R_{11}\theta_1 \approx -\begin{bmatrix} R_{1\kappa} & R_{12} \end{bmatrix} \begin{bmatrix} -1 \\ \hat{\theta}_2{}^* \end{bmatrix} \quad (20)$$

The pseudo-inverse of R_{11} is used to solve for θ_1 in a LS sense to give:

$$\hat{\theta}_1{}^* = -(R_{11}^T R_{11})^{-1} R_{11}^T \begin{bmatrix} R_{1\kappa} & R_{12} \end{bmatrix} \begin{bmatrix} -1 \\ \hat{\theta}_2{}^* \end{bmatrix} \quad (21)$$

The equalities of (17) can be used to express (21) as

$$\hat{\theta}_1{}^* = -R_{uu}^{-1} R_{uv} \begin{bmatrix} -1 \\ \hat{\theta}_2{}^* \end{bmatrix} \quad (22)$$

Clearly, considering (19) and (15), it must be true that $\hat{\theta}_1{}^*$ is equal to $\hat{\gamma}_1{}^*$.

3.4. Unbiased Property of Mixed LS-TLS

We have established that one approach to determining the mixed LS-TLS solution involves estimation of an eigenvector of the $R_{y/u}$ matrix. Given that the noise is uncorrelated with the input, we can see from (14) that the only term affected by the noise is R_{yy}. Furthermore, if the noise is white, R_{yy} differs from R_{vv} on only the main diagonal by an amount equal to the noise variance. Therefore, the eigenvector associated with the minimum eigenvalue extracted from R_{yy} (or a sum of R_{yy} and other noise-free components) can only differ from the eigenvector associated with the minimum eigenvalue of R_{vv} in magnitude and not direction. Since γ_2 is post-normalized, the magnitude of the eigenvector is inconsequential. The remainder of the parameter vector γ_1 is computed from noise-free quantities. Thus we conclude that the addition of white, uncorrelated noise does not bias the solution away from recovering the true parameter vector.

4. QR-RMTLS ALGORITHM

We now propose and describe a QR based recursive algorithm for mixed LS-TLS, dubbed QR-RMTLS. A QR decomposition is used to track the sample data matrix but an inverse iteration procedure is used to track the minimum eigenpair of $R_{y/u}$.

The extended data matrix C(k) is factored into its QR decomposition as $C(k) = Q(k)\overline{R}(k)$, where Q(k) is a unitary matrix and $\overline{R}(k)$ is upper triangular. $\overline{R}(k)$ is a tall matrix, with all but the first N+M+2 rows zero. For compactness sake, let R(k) represent the first N+M+2 rows of $\overline{R}(k)$.

For each new $\phi(k)$, R(k) is computed from R(k-1) via a transformation matrix T(k) as

$$\begin{bmatrix} R(k) \\ 0 \end{bmatrix} = T(k) \begin{bmatrix} R(k-1) \\ \phi(k) \end{bmatrix} \quad (23)$$

where T(k) is the culmination of a series of Givens rotations (see [3]). In this manner, we are able to track the R(k) matrix faithfully, efficiently and compactly.

It must be true that the resulting $\overline{R}(k)$ is a continuation of the partial QR decomposition expressed in (10). The full QR factorization can be expressed as

$$Q_{add}^T \cdot Q_{part}^T C(k) = \begin{bmatrix} R_{11}(k) & R_{1\kappa}(k) & R_{12}(k) \\ 0 & R_{\rho 1}(k) & R_{\rho 2}(k) \\ 0 & 0 & 0 \end{bmatrix} \quad (24)$$

We intend to use the sub-matrix $R_\rho(k) = [R_{\rho 1}(k)\ R_{\rho 2}(k)]$ to track the right hand singular vectors of $[R_{2\kappa}(k)\ R_{22}(k)]$. We therefore require that these two matrices have the same right hand singular vectors. By inspection, we claim that this is true since $\overline{R}(k)$ is calculated by a series of additional left sided orthogonal transformations. These transformations do not perturb the right handed singular vectors.

The eigenvector associated with the minimum eigenvalue of the matrix $R_\rho^T(k)R_\rho(k)$ ($\gamma_2(k)$) is tracked via a two-step inverse iteration procedure. The procedure is best illustrated via the following pseudo-code:

```
set γ2(k) = γ2(k – 1)
iterate until convergence is detected:
solve for ζ in RρT(k)ζ = γ2(k)
solve for ξ in Rρ(k)ξ = ζ
set γ̂2(k) = –ξ/ξ1 (impose monic constraint)
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Note that in the above procedure, $R_\rho(k)$ is upper triangular (and hence $R_\rho^T(k)$ is also triangular). Therefore, no matrix inversions are required to compute the desired vectors; back substitution is instead employed to simplify the computations. Furthermore, since the iteration begins with the previous estimate of a slowly changing process, one iteration pass is deemed sufficient.

The remainder of the coefficient vector estimate $\gamma_1(k)$ can be found from (20) by solving:

$$R_{11}(k)\hat{\gamma}_1(k) = -\left[R_{1\kappa}(k)\ R_{12}(k)\right]\hat{\gamma}_2(k) \qquad (25)$$

Note that in the above, $R_{11}(k)$ is also upper triangular; thus (25) can also be solved via back substitution, avoiding the computational expense of matrix inversion.

The following is a summary of the algorithm.

Set: $R(0) = I$, $\gamma(0) = (1/(N+M+2))\left[1\ \dots\ 1\right]^T$
for each k:
 Compute R(k) from R(k-1) and $\phi(k)$ via (23)
 Perform one-step back substitution inverse iteration procedure using $R_\rho(k)$ and $\gamma_2(k-1)$ to give $\hat{\gamma}_2(k)$
 Solve (25) using $\hat{\gamma}_2(k)$ and sub-matrices of R(k) via back substitution to give $\hat{\gamma}_1(k)$

5. QR-RMTLS ASYMPTOTIC BEHAVIOR

Clearly, as the iteration get large, (16) must converge to the true correlation matrix. Therefore, the QR decomposition of the extended data matrix must also asymptotically converge such that (18) is true. Given that $R_\rho(k) \to R_\rho$ as $k \to \infty$, the inverse iteration procedure using the previous estimate acting upon a constant matrix will also converge. Clearly, the denominator polynomial estimate γ_2 will converge to the eigenvector associated with the minimum eigenvalue of the matrix $R_\rho^T R_\rho$. Therefore, since $\hat{\gamma}_2$ has converged and is unbiased (under the stated noise conditions), it must be true that γ_1 has also converged and is unbiased. Thus, we have argued that in the asymptotic case under white uncorrelated noise, QR-RMTLS converges to the mixed LS-TLS solution.

6. SIMULATION RESULT

Shown in Figure 2 is a comparison of the normalized parameter error of the QR-RMTLS algorithm versus QR-RLS. The input sequence {u(k)} is a unit-variance white sequence. The output noise is white and uncorrelated with the input, with variance such that the SNR is 6 dB. The true system is a second order section with

$$a = \left[1\ 0.5\ 0.5\right]^T \text{ and } b = \left[0.5\ -0.5\ 0.25\right]^T \qquad (26)$$

As can be seen from Figure 2, QR-RMTLS is much preferred over QR-RLS.

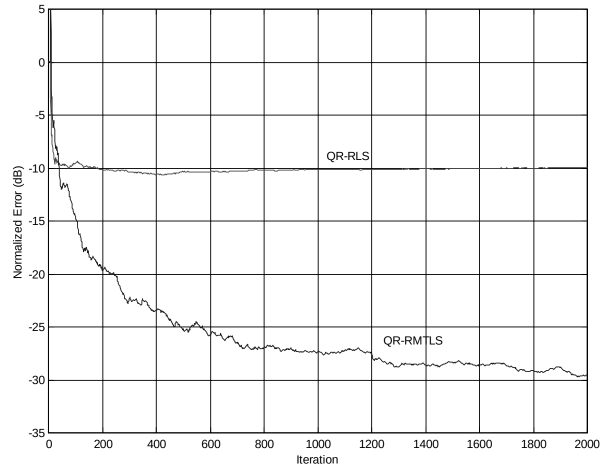

Figure 2: QR-RMTLS Simulation Performance

7. CONCLUSION

We have presented an iterative algorithm for unbiased equation error filtering by applying the mixed LS-TLS technique. Furthermore, we have presented an efficient implementation based on a QR decomposition and inverse iteration via back substitution. We then argued that the algorithm will converge asymptotically to the desired solution, and we demonstrated its performance via simulation.

8. REFERENCES

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