Parallelizable Eigenvalue Decomposition Techniques via the Matrix Sector Function

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Abstract

Many modern high-resolution spectral estimators in signal processing and control make use of the subspace information afforded by the singular value decomposition of the data matrix, or the eigenvalue decomposition of the covariance matrix. The derivation of these estimators involves some form of matrix decomposition. In this paper, new computational techniques for obtaining eigenvalues and eigenvectors of a square matrix are presented. These techniques are based on the matrix sector function which can be applied to break down a given matrix into matrices of smaller dimensions and consequently this approach is suitable for parallel implementation. Finally, an example which illustrates the proposed method is provided.

1 Introduction

Signal processing based on the eigenstructure of subspaces is the subject of intense research because it can be used in sensor array and time series problems to effectively increase the signal-to-noise ratio (SNR) and thereby enhance performance. Performance enhancement typically involves increased resolution, reduced variance of estimated parameters, and the extension of threshold breakdowns to lower SNR. In many cases, theoretical performance bounds are closely approached when subspace processing is applied. Many of these signal subspace approaches depend on the eigenvalue decomposition of an estimated covariance matrix.

In subspace methods, the autocorrelation matrix of large order (usually much bigger than the number of sources to provide a more accurate modeling of the noise) obtained from the data is decomposed and the signal and noise vectors are extracted according to the relative magnitudes of the singular values. Pisarenko [1] was one of the firsts to apply eigenanalysis to the problem of extracting signal information from an estimated at a covariance matrix. In recent years, signal subspace approaches have been applied to obtain high resolution estimators which include estimating the direction of arrival (DOA) and the sinusoidal frequency estimation problems, [2]-[4]. However, eigen-decomposition methods are computationally demanding since they involve the computation of each singular eigenvector and corresponding eigenvalue. In high resolution methods such as MUSIC, Minimum Norm, and ESPRIT, the noise and/or signal subspace are all that needed rather than the individual singular vectors. To reduce the computational cost associated with these methods, various alternatives were proposed by several authors. Kay and Shaw [5] suggested the use of polynomials and rational functions of the sample covariance matrix for approximating the signal subspace. In [6], Tufts and Melissinos used Lanczos and power-type methods to approximate the signal subspace. Karhunen and Joutsenalo [7] approximated the signal subspace using the discrete Fourier and Cosine transforms. Ermolaev and Gershman [8] used powers of sample covariance matrix based on Krylov subspaces to approximate the noise subspace. For useful articles and books, the reader is referred to [9], [10]-[12] and the references therein.

There are several algorithms for computing eigenvalues and eigenvectors. Most popular are the power method, the QR and LR algorithms [13]-[16]. The methods presented here are based on calculating the matrix sector function of a given matrix iteratively from blocks of eigenvalues and eigenvectors projected in specific regions of the complex plane. The basic methodology is developed by mapping different groups of eigenvalues of a given matrix onto a smaller known set of distinct complex numbers assigned to each sector in the complex plane.

In this paper, we propose fast convergent algorithms for computing an invariant subspace of Hermitian and non Hermitian matrices. Once we have block diagonalized a matrix, one can recursively apply these algorithms to each of the diagonal blocks. This in effect gives algorithms for computing the eigenvalue decomposition of any matrix.

2 Definition and Properties of the Matrix Sector Function

Let $A \in \mathcal{C}^{m\times m}$ be a nonsingular complex matrix with no negative eigenvalue, where $\mathcal{C}$ is the field of complex numbers. The matrix sector function of a matrix A, denoted by $S_n(A)$, is defined as

$$S_n(A) = A(\sqrt[n]{A^n})^{-1}, \tag{1}$$

where $\sqrt[n]{A^n}$ is the principal nth root of A^n. The principal nth root of the complex matrix A is defined to be any matrix $B \in \mathcal{C}^{m\times m}$ such that $B^n = A$ and for every $\gamma_r \in \sigma(B)$, $\sigma(A)$ denotes the set of eigenvalues of A, we have $\gamma_r = |\gamma_r|e^{i\theta_r}$, where $\theta_r \in (-\pi/n, \pi/n)$, for $r = 1, \cdots, m$ and i is the complex number $\sqrt{-1}$.

Generally, an nth root of a complex matrix $A \in \mathcal{C}^{m\times m}$ is defined to be any matrix $X \in \mathcal{C}^{m\times m}$ such that $X^n =$

A. When all eigenvalues of a non singular matrix A are distinct there are n^m distinct nth roots of A since if $A = P^{-1}\text{diag}(\lambda_1, \cdots, \lambda_m)P$, then each of the matrices $P^{-1}\text{diag}(w^{j_1}\sqrt[n]{\lambda_1}, \cdots, w^{j_m}\sqrt[n]{\lambda_m})P$ is also an nth root of A, for every set of integers $\{j_1, j_2, \cdots, j_m\} \subset \{0, 1, \cdots, n-1\}$, where ω is a primitive nth root of unity.

From the above introduction we observe that the matrix sector function is an nth root of the identity matrix I_m which commutes with A, i.e., $S_n(A)^n = I_m$, and $AS_n(A)^{-1} = S_n(A)^{-1}A$ has all of its eigenvalues in the sector $-\frac{\pi}{n} < \theta < \frac{\pi}{n}$. These two conditions can be viewed as a characterization of the matrix sector function. Note that for the matrix sign function this implies that $S_2(A)^2 = I_m$, and that $S_2(A)A$ has all its eigenvalues in the right half plane.

The matrix sector function provides an elegant way of splitting $\mathcal{C}^n$ into many complementary subspaces without actually computing any eigenvalues. The matrix $S_n(A)$ is diagonalizable and has the same invariant subspaces as A; its eigenvalues are nth roots of unity corresponding to eigenvectors of A whose eigenvalues are in the sectors $\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}$, $k = 0, 1, \cdots, n-1$. The region $\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}$ of the complex plane will be called the "kth sector". The sector $\frac{-\pi}{n} < \theta < \frac{\pi}{n}$ will be called the principal sector. Throughout this paper, the notation $\lambda(A)$ will be used to denote an eigenvalue of A.

From the definition (1) we can state several important properties of the matrix sector function.

Theorem 1. *the matrix sector function $S_n(A)$ satisfies the following properties:*

(a) $S_n(A^T) = S_n(A)^T$ *and* $S_n(A^*) = S_n(A)^*$.

(b) $S_n(\alpha A) = S_n(\alpha)S_n(A)$, *where α is any nonzero complex number such that* $\frac{-\pi}{n} \neq arg(\alpha) \neq \frac{\pi}{n}$.

(c) $S_n(A)A = AS_n(A)$.

(d) *The eigenvalues of $S_n(A)$ are nth roots of 1, i.e.,* $S_n(A)^n = I_m$.

(e) $AS_n(A)^{-1} = S_n(A)^{-1}A$ *and all eigenvalues of* $AS_n(A)^{-1}$ *are in the sector* $(\frac{-\pi}{n}, \frac{\pi}{n})$.

(f) *If V and Z are nonsingular matrices of same order, then* $S_n(V^{-1}ZV) = V^{-1}S_n(Z)V$, *provided that $S_n(Z)$ is defined.*

(g) $S_n(w^r A) = w^r S_n(A)$, *where w is any primitive nth root of 1.*

(h) $S_n(A^{-1}) = S_n(A)^{-1}$.

Proof: the proof follows directly from (1).

3 Eigen-Decomposition Using the Matrix Sector Function

In this section, we propose a method for computing invariant subspace decomposition. These methods exploit the notion of the matrix sector function resulting in methods of higher order convergence. These higher order methods are derived by mapping different groups of eigenvalues to a smaller known set which consists of distinct complex numbers assigned to each sector in the complex plane. To illustrate the proposed procedure, let $A \in \mathcal{C}^{m\times m}$. The Jordan canonical form of A can be expressed as

$$A = PI_AP^{-1}, \tag{2}$$

where I_A is a triangular matrix of Jordan blocks of the eigenvalues and P is a transformation matrix containing the associated eigenvectors. Clearly, if A is diagonalizable then I_A is a diagonal matrix. Now, from Theorem 1 the matrix sector function of A can be expressed as

$$S_n(A) = PS_n(I_A)P^{-1}, \tag{3}$$

where $S_n(I_A)$ is a diagonal matrix with eigenvalues, $1, w, w^2, \cdots w^{n-1}$ with $w^n = 1$. Note that $S_n(A)$ and A have the same eigenvectors. If an eigenvalue of the matrix A is in the i-sector, then the corresponding eigenvalue of $S_n(A)$ is w^{i-1}. Now assume that A is diagonalizable and that $S_n(I_A) = \text{diag}(I_{m_1}, wI_{m_2}, \cdots, w^{n-1}I_{m_n})$, where m_l represents the number of eigenvalues of A in the $(l-1)$-sector, then

$$S_n(A) = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \cdots & \cdots & \cdots & \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \text{diag}(I_{m_1}, wI_{m_2}, \cdots, w^{n-1}I_{m_n}) \times \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \cdots & \cdots & \cdots & \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}^{-1}, \tag{4}$$

where $P = [P_{ij}]$ is partitioned into blocks of compatible dimensions.

The following theorem is the cornerstone of parallel computation of EVD via the matrix sector function.

Theorem 2. *Let P and $S_n(A)$ be partitioned into blocks as described above and assume that the matrices $\{P_{ii}\}_{i=1}^n$ are invertible, then*

$$R = I_A^{n-1}P + PI_A^{n-1} + I_A^{n-2}PI_A + I_API_A^{n-2} + I_A^{n-3}PI_A^2 + I_A^2PI_A^{n-3} + \cdots + I_A^{\lfloor\frac{n}{2}\rfloor}PI_A^{\lfloor\frac{n}{2}\rfloor}, \tag{5}$$

is block diagonal given by $R = n\,\text{diag}(P_{11}, wP_{22}, \cdots, w^{n-1}P_{nn})$. Let $V = P^{-1}R$, then

$$V^{-1}AV = \text{diag}(P_{11}^{-1}D_{11}P_{11}, P_{22}^{-1}D_{22}P_{22}, \cdots, P_{nn}^{-1}D_{nn}P_{nn}) = \text{diag}(A_i), \tag{6}$$

where $I_A = \text{diag}(D_{11}, D_{22}, \cdots, D_{m_nm_n})$, and $A_i = P_{ii}^{-1}D_{ii}P_{ii}$ and $\lfloor x\rfloor$ denotes the largest integer less than or equal to x.

Proof: It can easily be shown by induction that R is block diagonal. The main conclusion follows from the following equation

$$V^{-1}AV = (P^{-1}R)^{-1}AP^{-1}R = R^{-1}P^{-1}APR = R^{-1}DR, \tag{7}$$

and the fact that the product of block diagonal matrices is also block diagonal. Q.E.D.

It should be noted that each of the matrices $P_{ii}^{-1}D_{ii}P_{ii}$ have the same eigenvalues as the matrix A_i and all lie in the same sector. Thus, the spectrum of A are split according to their position in the sectors. This process can be continued if necessary to split the spectrum of the blocks A_i. The computation can be continued in parallel until all submatrices reduce to Jordan form. An example showing how to apply Theorem 2 is presented next.

3.1 EVD Using the Matrix Sign Function

Let us next illustrate the idea of utilizing the matrix sector function S_2 for the parallel computation of EVD, let A, I_A, and P be as in Theorem 2. In this case, $R = I_A P + P I_A$, and therefore $V = P^{-1}R = P^{-1}I_A P + I_A$. To compute V, we need to compute $S_2(A)$ and I_A first. The dimensions of the Jordan blocks of I_A are unknown and can be computed as follows. Assume that the two Jordan blocks are of dimensions m_1 and m_2, then $S_2(A)$ is similar to $\text{diag}(I_{m_1}, -I_{m_2})$. Clearly,

$$\text{trace}(S_2(A)) = m_1 - m_2,$$
$$n = m_1 + m_2.$$

Solving for m_1, and m_2 yields

$$m_1 = \frac{n + \text{trace}(S_2(A))}{2},$$
$$m_2 = \frac{n - \text{trace}(S_2(A)}{2}.$$

Here $\text{trace}(B)$ denotes the sum of diagonal elements of B. Thus the matrix V which block diagonalizes A is completely determined.

3.2 EVD Using the Third Order Matrix Sector Function

Let us next illustrate the idea of utilizing the matrix sector function $S_3(A)$ for the parallel computation of EVD, let A, I_A, and P be as in Theorem 2. In this case

$$R = I_A^2 P + P I_A^2 + I_A P I_A,$$

and therefore

$$V = P^{-1}R = P^{-1}I_A^2 P + I_A^2 + P^{-1}I_A P I_A.$$

To compute V, we need to compute $S_3(A)$ and I_A first. The dimensions of the Jordan blocks of I_A are unknown and can be computed as follows. Assume that the three Jordan blocks are of dimensions m_1, m_2, and m_3, then $S_3(A)$ is similar to $\text{diag}(I_{m_1}, wI_{m_2}, w^2 I_{m_3})$, where $w \neq 1$ is a cubic root of 1. Additionally one can show that

$$\begin{aligned} \text{trace}(S_3(A)) &= m_1 + w m_2 + w^2 m_3, \\ \text{trace}(S_3(A)^2) &= m_1 + w^2 m_2 + w m_3, \\ n &= m_1 + m_2 + m_3. \end{aligned} \tag{8a}$$

Solving for m_1, m_2, and m_3 yields

$$\begin{aligned} m_1 &= \frac{n + \text{trace}(S_3(A)) + \text{trace}(S_3(A))^2}{3}, \\ m_2 &= \frac{n + w\,\text{trace}(S_3(A)) + w^2\,\text{trace}(S_3(A))^2}{3}, \\ m_3 &= \frac{n + w^2\,\text{trace}(S_3(A)) + w\,\text{trace}(S_3(A))^2}{3}. \end{aligned} \tag{8b}$$

Thus the matrix $V = I_A^2 + S_3(A)I_A + S_3(A)^2$, which block diagonalizes A, is completely determined.

4 Computation of $S_n(A)$

As indicated in the previous section, invariant subspace decomposition via the proposed approach requires the computation of the matrix sector function. For Theorem 2 to be applicable, one must develop efficient methods for computing the matrix sector function. The matrix sector function can be computed as follows:

Theorem 3 [17]. *Let A be a nonsingular $m \times m$ matrix such that none of its eigenvalues are on the boundary of the sectors $\{\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}\}_{k=0}^{n-1}$. Then the matrix sector function has the following integral representations:*

$$S_n(A)^{-1} = \frac{n\sin(\frac{\pi}{n})}{\pi} \int_0^\infty (y^n I_m + A^n)^{-1} A^{n-1} dy, \tag{9a}$$

and

$$S_n(A) = \frac{n\sin(\frac{\pi}{n})}{\pi} \int_0^\infty (y^n A^n + I_m)^{-1} A dy. \tag{9b}$$

Part (9b) of this result follows directly from (9a) and the relation $S_n(A^{-1}) = S_n(A)^{-1}$.

The integrals (9) can be computed over the finite interval $[0, \frac{\pi}{2}]$ by using the change of variable $y = \tan(\theta)$ in which case (9) transcribed to

$$S_n(A)^{-1} = \frac{n\sin(\frac{\pi}{n})}{\pi} \times \int_0^{\frac{\pi}{2}} (\sin^n(\theta) I_m + A^n \cos^n(\theta))^{-1} A^{n-1} \cos^{n-2}(\theta) d\theta, \tag{10a}$$

and

$$S_n(A) = \frac{n\sin(\frac{\pi}{n})}{\pi} \int_0^{\frac{\pi}{2}} (A^n \sin^n(\theta) + \cos^n(\theta) I_m)^{-1} A \cos^{n-2}(\theta) d\theta. \tag{10b}$$

In the next example we illustrate how to apply Theorem 2 and the matrix sector function to compute the matrix eigendecomposition.

5 Example

Consider the following 6×6 complex hermitian matrix A. To fit this matrix in a single column, it is written as $A = [A_1 : A_2]$, where A_1 and A_2 are 6×3 complex matrices representing the the first and last three columns of A, respectively. The eigenvalues of A are given by the set $\{5.3375, 1.4203, -3.0970, -1.6982, -1.1247, -0.8379\}$. The matrices A_1 and A_2 are given as:

$$A_1 = \begin{bmatrix} -0.1101 & 1.1454 - 0.1235i & 0.2044 - 0.4476i \\ 1.1454 + 0.1235i & -0.7610 & 1.3670 + 0.3226i \\ 0.2044 + 0.4476i & 1.3670 - 0.3226i & 0.8186 \\ 1.2281 + 0.4249i & 1.7474 - 0.5630i & 0.6553 - 0.3940i \\ 0.7005 + 0.8174i & 1.8891 + 0.0217i & 0.5996 - 0.7140i \\ 0.6686 - 0.3036i & 1.3173 - 0.5199i & 1.0431 + 0.5153i \end{bmatrix},$$

and

$$A_2 = \begin{bmatrix} 1.2281 - 0.4249i & 0.7005 - 0.8174i & 0.6686 + 0.3036i \\ 1.7474 + 0.5630i & 1.8891 - 0.0217i & 1.3173 + 0.5199i \\ 0.6553 + 0.3940i & 0.5996 + 0.7140i & 1.0431 - 0.5153i \\ -0.2250 & 1.4371 - 0.0381i & 0.6843 + 0.6344i \\ 1.4371 + 0.0381i & 0.0726 & 0.4183 + 0.0213i \\ 0.6843 - 0.6344i & 0.4183 - 0.0213i & 0.2049 \end{bmatrix}.$$

If the integral formula (10b) is applied using Gaussian quadrature with 16 points we obtain an approximation for $S_2(A)$ such that $||(S_2(A)^2 - I_6)||_2 = 1.3819(10)^{-7}$. The computed matrix sign function is given by $S_2(A) = [B_1 : B_2]$, where

$$B_1 = \begin{bmatrix} -0.5279 - 0.0000i & 0.3259 - 0.0996i & -0.2183 - 0.0352i \\ 0.3259 + 0.0996i & -0.5037 + 0.0000i & 0.3345 + 0.1099i \\ -0.2183 + 0.0352i & 0.3345 - 0.1099i & 0.1458 + 0.0000i \\ 0.4298 + 0.1661i & 0.4296 - 0.0369i & 0.0442 - 0.1547i \\ 0.3198 + 0.3316i & 0.3714 - 0.0104i & 0.1546 - 0.4474i \\ 0.1570 - 0.3270i & 0.3877 - 0.1786i & 0.5101 + 0.5439i \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.4298 - 0.1661i & 0.3198 - 0.3316i & 0.1570 + 0.3270i \\ 0.4296 + 0.0369i & 0.3714 + 0.0104i & 0.3877 + 0.1786i \\ 0.0442 + 0.1547i & 0.1546 + 0.4474i & 0.5101 - 0.5439i \\ -0.4971 - 0.0000i & 0.4812 - 0.1418i & 0.1445 + 0.2369i \\ 0.4812 + 0.1418i & -0.4063 + 0.0000i & -0.0664 + 0.0668i \\ 0.1445 - 0.2369i & -0.0664 - 0.0668i & -0.2106 - 0.0000i \end{bmatrix}.$$

The matrix I_A can be shown to be $I_A = \begin{bmatrix} I_2 & 0 \\ 0 & -I_4 \end{bmatrix}$. Therefore the matrix V is given by $V = I_A + S_2(A)$. The matrix V block diagonalizes A so that $VAV^{-1} = [C_1 : C_2]$, where

$$C_1 = \begin{bmatrix} 1.5957 - 0.7318i & -0.2685 - 1.0322i & 0.0000 - 0.0000i \\ 2.0870 + 1.6976i & 5.1621 + 0.7318i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 - 0.0000i & -1.7049 - 0.5310i \\ 0.0000 + 0.0000i & 0.0000 - 0.0000i & -0.2739 - 0.0132i \\ 0.0000 + 0.0000i & 0.0000 - 0.0000i & -0.6220 + 0.0334i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & -0.1006 - 0.3769i \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.0000 - 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ -0.7852 - 0.4513i & -1.7766 + 0.4180i & -0.5034 - 0.0854i \\ -1.9547 - 0.0818i & -0.0664 + 0.6053i & -0.2183 + 0.2803i \\ -0.3645 - 0.1728i & -1.6446 + 0.7817i & -0.2339 - 0.0767i \\ -0.3651 - 0.7337i & -0.8378 - 0.2098i & -1.4537 - 0.1689i \end{bmatrix}.$$

Note that the matrix VAV^{-1} is block diagonal. This process can be repeated in parallel if necessary untill all blocks reduce to one dimensional matrices.

6 Conclusion

In this paper, the main objective is to develop fast parallelizable matrix inverse free algorithms for the general algebraic eigenvalue problem using the matrix sector function. The resulting methods are highly modular and well suited for parallel implementation. The basic principle of this approach can be used to compute the complete EVD of any matrix. However, the efficiency of this technique is mainly influenced by the accuracy of computing the matrix sector function. Using the integral representation given in (9)-(10) and [17], the nth order matrix sector function can be computed using the Pade approximation from which exact partial fraction expansion can be obtained. Parallel implementation for the special case $n = 2$ is given in [11]. This novel concept will not only be useful as a powerful tool in signal/image processing, and control theory but also as a new parallelizable approach in computational linear algebra.

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