RATE DISTORTION OPTIMAL SIGNAL COMPRESSION USING SECOND ORDER POLYNOMIAL APPROXIMATION

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ABSTRACT

In this paper we present a time domain signal compression algorithm based on the coding of line segments which are used to approximate the signal. These segments are fit in a way that is optimal in the rate distortion sense. The approach is applicable to many types of signals, but in this paper we focus on the compression of ElectroCardioGram (ECG) signals. As opposed to traditional time-domain algorithms, where heuristics are used to extract representative signal samples from the original signal, an optimization algorithm is formulated in [1, 2, 3] for sample selection using graph theory, with linear interpolation applied to the reconstruction of the signal. In this paper the algorithm in [1, 2, 3] is generalized by using second order polynomial interpolation for the reconstruction of the signal from the extracted signal samples. The polynomials are fitted in a way that guarantees minimum reconstruction error given an upper bound on the number of bits. The method achieves good performance compared both to the case where linear interpolation is used in reconstruction of the signal and to other state-of-the-art ECG coders.

1. INTRODUCTION

An ECG signal is a graphic tracing of the variations in electrical potential caused by the excitation of the heart muscle and detected at the body surface. Such signals are often subject to transmission or long time storage. This calls for their efficient compression in order to keep their sizes manageable.

Time domain methods for ECG compression are based on the idea of extracting a subset of *significant* signal samples from the original sample set to represent the signal. Which signal samples are significant, depends on the underlying criterion for the sample selection process. To get a high performance time-domain compression algorithm, much effort should be put into designing intelligent sample selection criteria. Decoding is based on interpolating this subset of samples. There exist quite a few time domain compression algorithms for ECG signals. A common characteristic of most of them is that they are based on fast heuristics in the sample selection process, at the expense of optimality. Examples of such algorithms are the popular FAN algorithm [4], the well known AZTEC algorithm [5] and recent attempts in improving time domain algorithms, such as SLOPE [6] and AZTDIS [7].

Recently, optimization methods for ECG compression were developed such as the *Operational Rate Distortion and Variable Length Coder* (ORD-VLC) optimal approach presented in [1, 2, 3]. This method is based on a rigorous mathematical model of the entire sample selection process. By modeling the signal samples as nodes in a graph, ECG signals are compressed by using known optimization theory to minimize the distortion in the reconstructed signal given an upper bound on the available number of bits. The approaches in [1, 2, 3] applied linear interpolation for the reconstruction of the signal. This is a simple, but computationally effective way of reconstructing the signal. However, segments of an ECG signal are not linear in their nature, but rather contain higher frequencies. It is therefore interesting to investigate if better approximations to the original signal can be obtained for the same compression ratio, using a polynomial of higher degree for interpolation. In this paper we demonstrate how the algorithm in [1, 2, 3] can be extended in order to reconstruct the signal by second order polynomials. The resulting solution is optimal in the operational rate distortion (ORD) sense. Given the structure of the coder, no other technique based on second order polynomial interpolation will give a lower distortion for the same bit rate. In addition, we apply an iterative procedure to find the underlying parameter probability distribution resulting in the locally most efficient ORD curve. Similar techniques, with and without VLC optimization, have been used for compression of image contours [8, 9, 10, 11, 12].

This paper is organized as follows: In the next section the problem is defined mathematically. Section 3 is devoted to the solution method. Finally, experimental results are reported, and in the concluding section different aspects of the method, along with future work, are discussed.

2. PROBLEM FORMULATION

Let us denote the set of *sample points* taken from a signal at constant time intervals by $S = \{(1, y(1)), \ldots, (N, y(N))\}$. Let us define the set of *admissible points* by $Y = \{(n, y(n, l)), n = 1, \ldots, N; l = -p, \ldots, p\}$, where $y(n, -p), \ldots, y(n, p)$ are evenly distributed signal values with $y(n, 0) = y(n)$, and let us denote the cardinality of Y by $N_Y = N(2p + 1)$. That is, Y is an extension of S representing all the points which can be used for the signal approximation, and $S \subseteq Y$. Finally, let $y(n, 1) - y(n, 0) = \Delta_Q$, the quantization step size of the quantizer applied to the original signal. We seek a compression set $C = \{n_1, \ldots, n_M\} \subseteq \{1, \ldots, N\}$, the cardinality M of C, as well as, for each n_i an

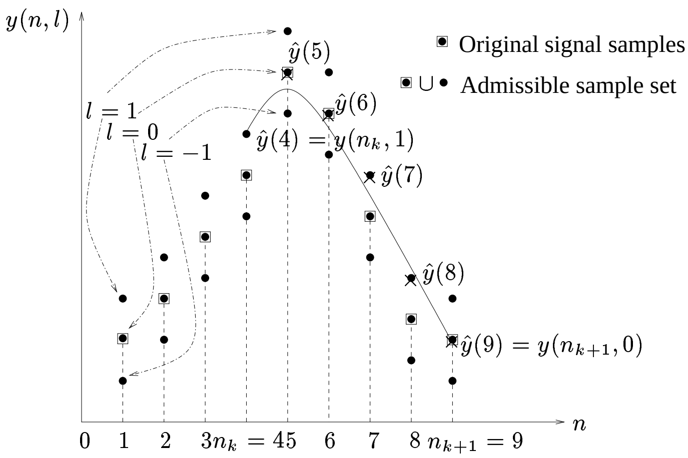

Fig. 1. The original signal and the admissible sample set.

integer $l_i \in \{-p,\ldots,p\}, i=1,\ldots,M$. The above definitions result in the original sample set S being replaced by the set $\hat{S}=\{(n_k, y(n_k,l_k)), k=1,\ldots,M\}$ implying a sample reduction ratio equal to N/M.

To guide the selection of C and $l_i, i=1,\ldots,l_M$, we assume a signal reconstruction based on second order polynomials interpolating $\hat{S}$. Assume $n_1=1$ and $n_M=N$. The approximation is then given by $\hat{y}(n)=y(n,l)$ if $n\in C$ and $\hat{y}(n)=Q\left(f_{n_k l_k n_{k+1} l_{k+1}}(n)\right)$, where $n_k<n<n_{k+1}$ for all $n\notin C, n\in\{1,2,...,N\}$. Here $f_{n_k l_k n_{k+1} l_{k+1}}(n)$ denotes a presumed reconstruction of $y(n)$ based on $y(n_k,l_k)$, $y(n_{k+1},l_{k+1})$ and all the intermediate samples to be precisely defined later in this section, and Q denotes quantization to the nearest integer. Hence all samples in the time interval $[n_k,n_{k+1}]$ are represented by second order polynomial interpolation between $(n_k, y(n_k,l_k))$ and $(n_{k+1},\ y(n_{k+1},l_{k+1}))$. This is illustrated in Figure 1 where we approximate the samples at time indices $n=\{4,\ldots,9\}$ by the curve starting in $n_k=4$ and ending in $n_{k+1}=9$. In this case, $y(n_k,1)$ belongs to Y, but *not* to S, while $y(n_{k+1},0)$ belongs to both Y and S, which means that it is one of the original signal samples.

The distortion measure we apply is the sum of squared distances between the original and the reconstructed signals. For the segment between two arbitrary admissible sample points $(n, y(n,l))$, and $(n',y(n',l'))\in Y$, with $n<n'$, it is given by

$$d_{nln'l'}=\begin{cases}\sum_{t=n}^{n'-1}\left(\hat{y}(t)-y(t)\right)^2 & \text{if } n'<N,\\ \sum_{t=n}^{n'}\left(\hat{y}(t)-y(t)\right)^2 & \text{if } n'=N.\end{cases}\qquad(1)$$

We are using second order polynomials in reconstructing the signal. Thus $f_{nln'l'}(t)=a_{0nln'l'}+a_{1nln'l'}t+a_{2nln'l'}t^2, t\in[n,n']$. If we interpolate between two end points $\mathbf{i}=(n,l)$ and $\mathbf{j}=(n',l')$, we have for each arc $(\mathbf{i},\mathbf{j})$:

$$d_{\mathbf{ij}}=\begin{cases}\sum_{t=n}^{n'-1}\left(a_{0\mathbf{ij}}+a_{1\mathbf{ij}}t+a_{2\mathbf{ij}}t^2-y(t)\right)^2 & \text{if } n'<N,\\ \sum_{t=n}^{n'}\left(a_{0\mathbf{ij}}+a_{1\mathbf{ij}}t+a_{2\mathbf{ij}}t^2-y(t)\right)^2 & \text{if } n'=N.\end{cases}\qquad(2)$$

$$a_{0\mathbf{ij}}+a_{1\mathbf{ij}}n+a_{2\mathbf{ij}}n^2=y(n,l)\qquad(3)$$

$$a_{0\mathbf{ij}}+a_{1\mathbf{ij}}n'+a_{2\mathbf{ij}}n'^2=y(n',l').\qquad(4)$$

The optimal parameters $a_{0\mathbf{ij}}, a_{1\mathbf{ij}}$ and $a_{2\mathbf{ij}}$ are found by minimizing (2) under the constraints given in (3) and (4). By inserting these optimal parameters into (2), the minimal $d_{\mathbf{ij}}$ is found for each arc.

The segment distortion measure established so far will serve as a quality measure for parts of the signal. Based on the segment distortion, it is clear that the total distortion of the reconstructed signal, D_{tot}, is made up of the sum of the segment distortions of the segments included in the final solution, that is,

$$D_{tot}=\sum_{k=1}^{M-1} d_{n_k l_k n_{k+1} l_{k+1}},\quad n_k,n_{k+1}\in C.\qquad(5)$$

Using second order polynomials in reconstructing the signal, three parameters must be encoded for each retained sample of the signal. We choose to represent each segment of the signal by two amplitudes and one position coordinate. We apply a simple predictive encoding scheme and encode the first order difference of all parameters (first order DPCM), that is, each segment of the signal is represented by the three parameters $\delta_{n(k)}=n_k-n_{k-1}$, $\delta^1_{y(k)}=\hat{y}(\frac{n_{k-1}+n_k}{2})-\hat{y}(n_{k-1})$, and $\delta^2_{y(k)}=\hat{y}(n_k)-\hat{y}(\frac{n_{k-1}+n_k}{2}), k=2,3,\ldots,M$. In addition, we need to encode the absolute amplitude of the first point, $\hat{y}(n_1)$. We choose to encode these symbols by two different coders; $\delta^1_{y(k)}$ and $\delta^2_{y(k)}$ by one coder and $\delta_{n(k)}$ by another coder in this context.

Let us denote the number of bits needed to encode the segment between points (n,l) and (n',l') by $r_{nln'l'}$. The total bit rate, R, can then be expressed as

$$R=\sum_{k=1}^{M-1} r_{n_k l_k n_{k+1} l_{k+1}}.\qquad(6)$$

We are then faced with the following problem : Choose M, $n_1<n_2<\cdots<n_M$ and $l_1,\ldots,l_M\in\{-p,\ldots,p\}$ which minimize the distortion of the reconstructed signal, D_{tot}, under the constraint that the total bit rate, R, is less than the maximum allowable bit rate, R_{max}. That is,

$$\min_{\hat{S}\in Y} D_{tot},\ \text{ subject to }\ R\le R_{max}.\qquad(7)$$

3. SOLUTION METHOD

In order to be able to apply a shortest path solution scheme to our problem, we define the problem in terms of graph theory. We build a graph, G, from the admissible sample set. The graph is directed and is defined as $G=(V,A)$ where the *vertex set* $V=\{(n,l), n=1,\ldots,N, l=-p,\ldots,p\}$ and the *arc set* A contains vertex pairs $((n,l),(n',l'))$, where $(n,l),(n',l')\in V$, $n<n'$ and $l,l'\in\{-p,\ldots,p\}$ as described in Section 2. If $(n_1,l_1),(n_M,l_M)\in V$, the set $(n_1,l_1),\ldots,(n_M,l_M)$ is said to be a *path* from (n_1,l_1) to (n_M,l_M) in G if $(n_1,l_1),\ldots,(n_M,l_M)\in V$ are distinct vertices and $n_1<n_2<\cdots<n_M$. Let P denote a path from vertex (n_1,l_1) up to vertex (n_M,l_M). The cost of each arc $((n,l),(n',l'))\in A$ is denoted $w_{nln'l'}$. It is made up of a combination of distortion and bit rate and will be defined later in this section. The length of P is thus the sum of the costs of segments included in the path up to vertex (n_M,l_M). Defining the problem this way, we are looking for the shortest path from vertex (n_1,l_1) to vertex (N,l_M).

We apply a shortest path algorithm to the graph to solve the problem stated in (7). The algorithm is a modified version of Dijkstra's shortest path algorithm [13], where the modification consists of taking into account the fact that we are working with a *directed* acyclic graph. This occurs as a natural consequence of the data we are working with. The shortest path algorithm is thoroughly described in [2]. In order to be able to solve the problem given in Equation (7) efficiently and optimally we use the Lagrangian multiplier approach [14]. The basic idea behind the approach is to include the constraint into the objective function with a Lagrangian multiplier λ. This results in a Lagrangian cost function of the following form

$$J_\lambda^C = D^C + \lambda \cdot R^C, \qquad (8)$$

where λ is the Lagrange multiplier and the superscript C denotes that the expression is a function of the compression set C under consideration. Minimization of the expression given in Equation (8) is well suited to be performed with the shortest path algorithm.

It has been shown in [14, 15] that if there is a λ^* such that

$$C^* = \arg \min_C J_{\lambda^*}^C, \qquad (9)$$

and which leads to $R^{C^*} = R_{max}$, then C^* is also an optimal solution to (7). As λ sweeps from zero to infinity, the solution to (9) traces the convex hull of the operational rate distortion function, which is a nonincreasing function. Therefore, by solving the unconstrained problem (9) repeatedly for different λ's we can find the optimal solution to the constrained problem (7).

Having introduced the Lagrangian multiplier approach, the edge weight between any two graph vertices (n, l) and (n', l'), $n < n'$, $l, l' \in \{-p, \ldots, p\}$, is given by $w_{nln'l'} = d_{nln'l'} + \lambda \cdot r_{nln'l'}$.

Applying the shortest path algorithm with this definition of an edge weight, leads to the minimization of

$$\sum_{k=1}^{M-1} w_{n_k l_k n_{k+1} l_{k+1}}, \qquad n_k \in C, l_k \in [-p, p], \qquad (10)$$

and, hence, to an optimal solution to the unconstrained problem (9). By solving this unconstrained problem repeatedly for different λ's we can find the optimal solution to the constrained problem (7).

Our claim of optimality is clearly dependent on the chosen code structure, the width of the admissible sample point band, the size of our window restricting how far apart two consecutive points of C may be, and, to a great extent, on the VLC tables. If we base the algorithm on fixed VLC tables generated off line, this will clearly make the performance of the encoder signal dependent as it is hard to find *one* VLC table to match the characteristics of different ECG waveforms.

In our case we iterate on the VLC table as a part of the compression scheme as shown in Figure 2. For each λ we find the VLC tables matching the frequencies of the output symbols. To start out the iterative process, depicted in Figure 2, the proposed ORD optimal encoder processes the ECG signal with an initial fixed rate distortion tradeoff, λ, and initial probability mass functions for $(\delta_{y(k)}^1, \delta_{y(k)}^2 | VLC_{init}^1)$ and $(\delta_{n(k)} | VLC_{init}^2)$. Having encoded the input sequence at iteration t, based on the probability mass functions $f_1^t()$ and $f_2^t()$ we use the frequency of the output symbols to compute $f_i^{t+1}(), i = 1, 2$ and then use $f_i^{t+1}()$ as basis for the VLC tables in iteration $t+1$. The iterative process of Figure 2 stops when the cost improvement is less than ϵ. At this point an outer loop checks if the total bit rate R, is close enough to the target bit rate, R_{max}. If it is, the symbols are encoded by a variable length coder. If not, another guess for λ is made, and the process is repeated. It can be shown that cost C^t is a non-increasing function of iteration t, and, hence, the iterative process converges to a local minimum.

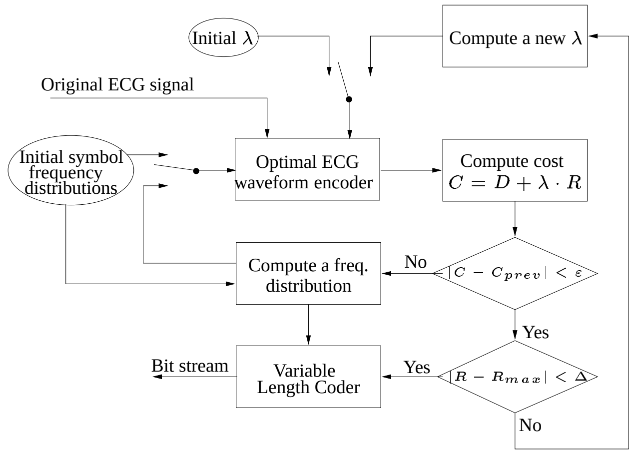

Fig. 2. The structure of the encoder.

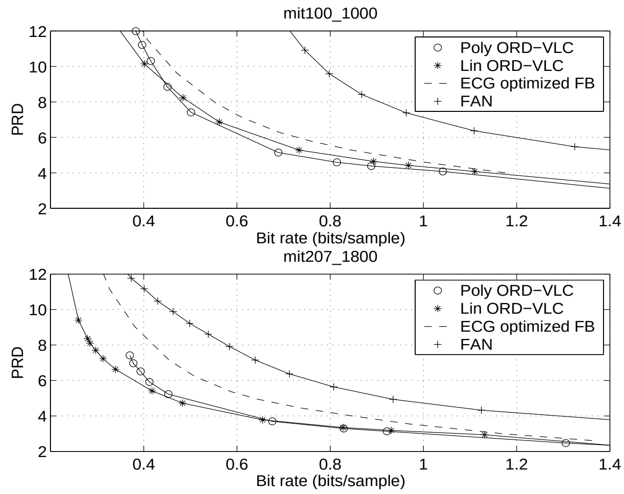

Fig. 3. Coding performance for the different coders. Solid line with circles: Poly ORD-VLC optimal approach with $p = 3$. Solid line with stars: Lin ORD-VLC optimal approach with $p = 3$. Dotted line: ECG optimized filter bank. Solid line with crosses: FAN

4. EXPERIMENTAL RESULTS

For evaluation of the performance of the coders, the commonly used *Percentage Root-mean-square Difference* distortion measure, given by

$$PRD = \sqrt{\frac{\sum_{l=1}^{N} [y(l) - \hat{y}(l)]^2}{\sum_{l=1}^{N_S} [y(l) - \bar{y}]^2}} \times 100\%, \qquad (11)$$

is applied, where $\overline{y}$ is the mean value of the original signal y, $\hat{y}$ is the reconstructed signal and N is the original signal length. We evaluate PRD as a function of *bit rate* which is defined as the average number of bits used to represent one signal sample in the original signal.

We present results for two test signals. The records are ten minutes long, corresponding to 216 000 samples. The sampling frequency is 360 Hz with 12 bits per sample.

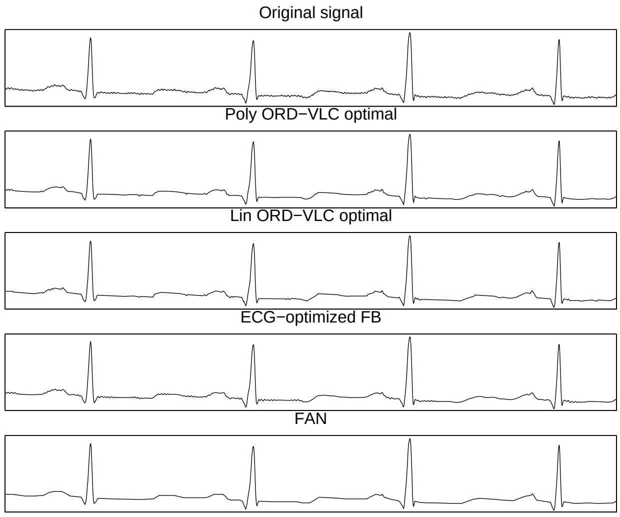

Fig. 4. Short segment of reconstructed signal (taken from mit100_1000) at 1.0 bits per sample.

Figure 3 presents PRD's for the various coders for bit rates between 0.2 and 1.4 bits per sample (bps). We refer to the method introduced in this paper as joint operational rate distortion and VLC (ORD-VLC) optimal approach. We distinguish between the method proposed in this paper and the method presented in [1, 2, 3] where straight lines are used in reconstruction of the signal by denoting them Poly ORD-VLC and Lin ORD-VLC, respectively. We compare the results from these two compression schemes to the results from the FAN algorithm [4], a traditional time domain ECG compression method, as well as a 32-tap ECG optimized filter bank. This is a parallel, nonunitary FIR filter bank optimized with respect to coding gain, as well as, visual criteria [16].

Results show that the FAN method is outperformed by a wide margin by all the other methods. The ORD-VLC methods outperform all the other methods by a significant margin, especially for low bit rates (below 0.8 bps). The Poly ORD-VLC method performs marginally better than the Lin ORD-VLC optimal approach for most bit rates.

Evaluation of the performance of the different coders is accompanied by visual inspection of the reconstructed signals. This is to show coding artifacts as they appear for the different coders. We have chosen a short segment of the mit100_1000 signal ("mit xxx_yyyy" denotes record number xxx starting at time yy:yy). The reconstructed signal is shown at a bit rate of 1.0 bits per sample in Figure 4. We see that all coders smooth out some of the details in the original signal. This is particularly evident with the FAN coder, where the line pieces are also most prominent in the reconstructed signal. The ORD-VLC optimal approaches smooth out the ripple noise which can be seen in the original signal.

5. CONCLUSIONS

In this paper we demonstrate how the rate distortion optimal approach presented in [1, 2, 3] can be further developed in order to apply second order polynomials in the reconstruction of the signal.

We compare the performance of the coder presented in this paper to the method presented in [1, 2, 3] in addition to a traditional ECG compression method as well as a state-of-the-art filter bank coder. Coding experiments show that the traditional time domain ECG compression algorithm cannot compete with the optimal operational rate distortion (ORD) coding techniques nor with the filter bank approach. The ORD coding techniques has superior performance compared to other time domain methods, as well as, frequency domain compression methods in terms of PRD. Compared to the algorithm in [1, 2, 3] where linear interpolation is applied in reconstruction of the signal, the algorithm presented here shows promising results. For most bit rates, the algorithm based on polynomial interpolation gives smaller reconstruction error than the one based on linear interpolation. This result is verified by visual inspection of the reconstructed signal.

6. REFERENCES

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