CHAOTIC AR(1) MODEL ESTIMATION

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ABSTRACT

Chaotic signals generated by iterating nonlinear difference equations may be useful models for many natural phenomena. In this paper we propose a family of chaotic models for signal processing applications. The chaotic signals generated by this family of first order difference equations have autocorrelations identical to stochastic first-order autoregressive (AR) processes. After considering the huge computational cost and the inconsistency of the optimal model estimator in the maximum-likelihood (ML) sense we propose low cost, suboptimal estimation approaches. Computer simulations show the good performance of the proposed modeling approach.

1. INTRODUCTION

Chaotic signals, signals generated by a non-linear dynamical system in chaotic state, have received much attention in the past years. Chaotic models have been proposed, for example, for sea clutter [1], speech waveforms [2], wind velocity fields [3], biomedical signals and in experimental physics, where many processes give rise to chaotic phenomena. However, chaotic modeling has several special properties. For example, the high sensitivity to initial conditions makes signal regeneration quite hard but, at the same time, it may be considered an advantage in representing anomalous behavior of signals over short periods of time [2].

The application of chaotic modeling is also conditioned by the lack of a family of chaotic models that combine a certain generality with easily computable estimation algorithms. Chaotic modeling with the Duffing equation has been considered in [2]. Neural networks as chaotic models have also been proposed in [1] and [3]. In these two cases the effect of noise is not considered and, in the case of neural networks, the models are impossible to analyze. Ideally, we would search for the chaotic equivalent of the ARMA models. To some extent, chaotic signals generated by piecewise-linear (PWL) maps of the unit interval could claim this title, since they have rational spectra [4].

Restricting the models to PWL maps on the unit interval allows the analysis of maximum likelihood (ML) signal estimators for a known map. The ML estimator is inconsistent, so the asymptotic distribution for large data records is invalid. However, for a high Signal to Noise Ratio (SNR), the ML estimator is asymptotically unbiased and attains the Cramer-Rao lower bound (CRLB) [5]. The ML estimator for chaotic signals generated by iterating known PWL maps is derived in [6]. Parameter estimation for chaotic systems has received much less attention, relying mostly on linear approaches, although ML estimators may be considered [7].

In this paper we develop parameter and signal estimators for a class of chaotic difference equations that produce signals with autocorrelations identical to those produced by stochastic first-order autoregressive processes. Our objective is to propose these models as the chaotic alternative to the AR(1) model. The numerical instability of the signal generation is avoided by generating the signal by backward iteration, where instead of error amplification [2], error reduction occurs. The exact ML model estimator is considered, but its computational cost and inconsistency makes it useless. Therefore, a combination of a prediction error minimization parameter estimation, by exploiting only the dependence between one sample and the next, with a ML signal estimator, which produces a low computational cost approach with good performance, is considered.

2. CENTERED SKEW-TENT MAPS

The signals $x[n]$ that we consider in this work are generated according to

$$x[n+1] = F(x[n]) \tag{1}$$

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where $F(\cdot)$ is the so called centered skew-tent map

$$F(x) = \begin{cases} \frac{2(1+x)}{1+a} - 1 & x \leq a \\ \frac{2(1-x)}{1-a} - 1 & x > a \end{cases} \quad (2)$$

for some parameter $-1 < a \leq 1$.

This map produces sequences that are chaotic with invariant density $p(x)$ uniform in the range $[-1, 1]$, [8]. The phase space of non-linear maps can be divided in a collection of non-overlapping regions. If a symbol from a known alphabet is assigned to each of the regions, the dynamics of the map may be characterized by following the different regions that the map visits during its dynamical evolution. In the particular case of the centered skew-tent map, we divide the phase space in two regions $E_1 = [-1, a]$ and $E_2 = [a, 1]$ and we associate a symbol $s[n]$ to each $x[n]$ according to

$$s[n] = sign(x[n] - a) \quad (3)$$

This sign sequence $\mathbf{s} = s[0], ..., s[N-1]$, also called itinerary, can be considered a symbolic coding of the chaotic signal. As the centered skew-tent maps are onto, all the itineraries are possible, and therefore, there are 2^N regions in the phase space.

It is easy to verify that $F(x[n])$ can be expressed as

$$F(x[n]) = \frac{-2ax[n] + 1 + a^2 - 2|x[n] - a|}{b} \quad (4)$$

with $b = 1 - a^2$. Using the symbol $s[n]$

$$F(x[n]) = \frac{(1 + 2as[n] + a^2) - 2(s[n] + a)x[n]}{b} \quad (5)$$

Chaotic signals generated according to the previous model have interesting statistical properties, that make them the equivalent of AR(1) models. As the invariant density is uniform in $[-1, 1]$, the autocorrelation may be computed as

$$\begin{aligned} R_{xx}[m] &= E\left[x[n]x[n+m]\right] = \int_{-1}^{1} xF^m(x)p(x)dx = \\ &= \frac{1}{2}\int_{-1}^{a} xF^{m-1}\left(\frac{2(1+x)}{1+a} - 1\right)dx + \\ &+ \frac{1}{2}\int_{a}^{1} xF^{m-1}\left(\frac{2(1-x)}{1-a} - 1\right)dx \end{aligned}$$

The final autocorrelation function, obtained changing variables in both integrals and integrating, becomes

$$R_{xx}[m] = r_0 a^m \quad (6)$$

with $r_0 = 1/3$. Therefore, the parameter a has the same relation with the autocorrelation as in the case of AR(1) processes [8].

3. ESTIMATION OF AR(1) CHAOTIC MODELS

3.1. Problem Statement

The signal model we are considering is

$$y[n] = x[n] + w[n] \qquad n = 0, 1, \ldots, N \quad (7)$$

where $x[n]$ is generated using (2) by iterating some unknown $x[0] \in [-1, 1]$ according to (1) for some unknown parameter $-1 < a \leq 1$. $w[n]$ is a stationary, zero-mean, white Gaussian noise with variance σ^2, and represents all the additive effects that have distorted the chaotic signal. A moderately high SNR situation is assumed (at least 10 dB). Model estimation demands obtaining an estimate of parameter a and of the initial condition $x[0]$ to reproduce $x[n]$.

ML model estimation produces the initial condition and the parameter that minimize

$$J(x[0], a) = \sum_{n=0}^{N} \left(y[n] - F^n(x[0], a)\right)^2 \quad (8)$$

This problem as it is stated has not been solved yet. Nonetheless, ML estimators for chaotic signals generated by PWL maps with known parameter have been developed [6]. The main point is that the ML estimator is feasible, although of high computational cost. Minimizing (8) will demand the computation of 2^N estimates, and for each one a gradient descent algorithm should be applied on a highly complex cost function. Furthermore, the inconsistency of these kind of estimators will imply that the performance will saturate for very short registers [7].

3.2. Parameter estimation

Using the known structure of the autocorrelation we can consider an equivalent to the normal equations for this problem. Due to the additive noise, however, we should use the modified Yule-Walker equations, obtaining the following estimator

$$\hat{a} = \frac{\hat{r}_{xx}[2]}{\hat{r}_{xx}[1]} \quad (9)$$

where $\hat{r}_{xx}[k]$ is the usual estimator of the autocorrelation. This estimator shows similar performance in this case as in the estimation of AR(1) models. A better performance, however, may be obtained by exploiting the deterministic relation that exists between each sample and the next. Using (4) we can attempt the minimization of the sum of the error squares

$$J(a) = \sum_{n=0}^{N-1} e[n]^2 = \sum_{n=0}^{N-1} \left(y[n+1] - F(y[n])\right)^2 \quad (10)$$

This nonlinear minimization problem may be solved by gradient descent approaches (using Newton-Raphson for example). However, the dependence of the itinerary on the parameter a will hinder the performance of the estimator. The proposed alternative consists of decomposing the problem in a set of linear ones as a function of the itinerary. Denoting $d[i] = 2y[i] + s[i](y[i+1]-1)$, we can define

$$\begin{aligned}\mathbf{d_s} &= [d[0], d[1]..., d[N-1]]^T \\ \mathbf{z} &= [1+y[1], 1+y[2], ..., 1+y[N]]^T\end{aligned}$$

where $\mathbf{s}$ is the vector of the sign components, and the subscript $\mathbf{s}$ stresses the dependence of $\mathbf{d}$ with the itinerary. Thus the error vector is $\mathbf{e_s} = \mathbf{d_s} - \mathbf{z}a$ and the sum of the error squares becomes

$$J_{\mathbf{s}}(a) = ||\mathbf{e_s}||_2^2 = ||\mathbf{d_s} - \mathbf{z}a||_2^2 \tag{11}$$

where $||\cdot||_2^2$ is the squared Euclidean norm. Obtaining the least squares (LS) solution requires considering the 2^N possible itineraries and minimizing (11) for each one. For a known itinerary the LS estimate of a is

$$\hat{a}_{\mathbf{s}} = \left(\mathbf{z}^T\mathbf{z}\right)^{-1}\mathbf{z}^T\mathbf{d_s} = \mathbf{z}^{\#}\mathbf{d_s} \tag{12}$$

However, in a high SNR situation, it is reasonable to consider only the N+2 possible itineraries produced by sorting the data samples and dividing them in two continuous sets. Thus we will obtain a Hard-Censoring LS (HCLS) estimate of the itinerary $\mathbf{s}$, which is the one among the $N+2$ sign sequences that minimizes

$$J_{\mathbf{s}}(\hat{a}_{\mathbf{s}}) = ||(\mathbf{I} - \mathbf{z}E_{\mathbf{z}}^{-1}\mathbf{z}^T)\mathbf{d_s}||_2^2 = ||\mathbf{E}\mathbf{d_s}||_2^2 \tag{13}$$

where $E_{\mathbf{z}} = \mathbf{z}^T\mathbf{z}$ is the squared norm of $\mathbf{z}$. Note that $\mathbf{E}$ does not depend on $\mathbf{s}$. Finally,the HCLS parameter estimate is computed applying (12) using the itinerary that minimizes (13).

3.3. Signal Estimation

Once the parameter estimate has been obtained, we can apply the ML estimator in [6] to obtain the signal estimate. For a given sign sequence $\mathbf{s}$, and parameter a the ML estimate of $x[N]$ is the value which minimizes

$$J(x[N]) = \sum_{n=0}^{N-1}\left(y[n] - F_{\mathbf{s}}^{-(N-n)}(x[N])\right)^2 \tag{14}$$

In our approach the parameter a, and itinerary $\mathbf{s}$ will be the ones obtained from the parameter estimation algorithm. To obtain a closed form expression for the estimate of $x[N]$ we need an expression for $F^{-n}(\cdot)$

$$\begin{aligned}F_{\mathbf{s}}^{-n}(x[N]) &= \sum_{i=0}^{n-1}\frac{(-1)^i b^i S_{n-i-1}^N}{2^{i+1}S_n^N}k[N-n+i] + \\ &\quad + (-1)^n 2^{-n} b^n (S_n^N)^{-1} x[N]\end{aligned} \tag{15}$$

where $k[n] = (1 + 2as[n] + a^2)$, $S_0^n = 1$, and

$$S_i^n = \prod_{j=n-i}^{n-1}(s[j]+a) \qquad i = 1, \ldots, n \tag{16}$$

and the estimate for $x[N]$ will be given by

$$\hat{x}[N] = \frac{\sum_{n=0}^{N-1}\alpha[n](y[n]-\gamma[n])}{\sum_{n=0}^{N-1}(\alpha[n])^2} \tag{17}$$

where

$$\alpha[n] = \frac{(-1)^{N-n}b^{N-n}}{2^{N-n}S_{N-n}^N} \tag{18}$$

and

$$\gamma[n] = \sum_{i=0}^{N-n-1}(-1)^i 2^{-(i+1)} b^i \frac{S_{N-n-i-1}^N}{S_{N-n}^N} k[n+i] \tag{19}$$

Note that we estimate x[N] instead of x[0] to avoid the numerical instability characteristic of the generation of chaotic signals by forward iteration. The rest of the signal $x[0], \cdots, x[N-1]$ will be obtained by iterating backwards from $\hat{x}[N]$ using (15), with the sign sequence $\mathbf{s}$ obtained from the parameter estimation algorithm.

4. SIMULATION RESULTS

In this section we analyze the performance of the parameter and signal estimation algorithms. Concerning parameter estimation we compare the HCLS solution, the gradient descent approach, and the method based on the known structure of the autocorrelation. We have simulated 1000 cases for each parameter value with different SNR values. For each case a chaotic signal with $N = 100$ and random initial condition has been generated. From figure 1 it can be inferred that the HCLS improves the performance of the gradient descent algorithm, achieving both of them a considerably better performance than the autocorrelation based estimator.

In figure 2 we show the MSE obtained for different SNRs in the signal generation using the parameter and itinerary estimated with the HCLS technique to apply the ML signal estimator. The best performance of the modeling approach considered is achieved for low absolute values of the parameter a. Finally, figure 3 shows an AR(1) process and an AR(1) chaotic signal with the same parameter a. The figure shows the different modeling capabilities of both approaches, even though they share the same spectral density.

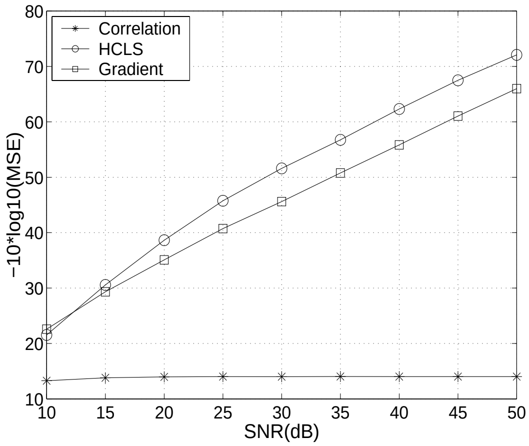

Figure 1: Comparison of different alternatives of parameter estimation for $N = 100$, $a = 0.5$.

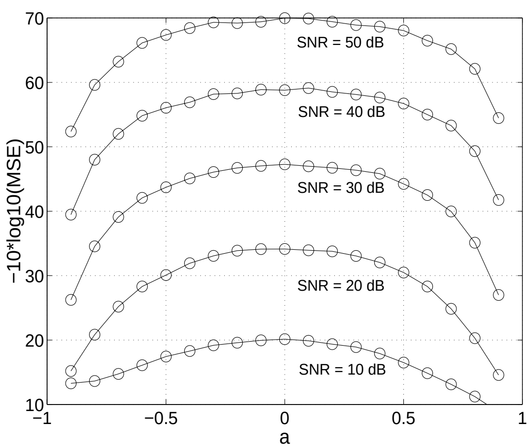

Figure 2: Comparison of the MSE of the model estimator (HCLS+ML) for $N = 100$, and different SNRs.

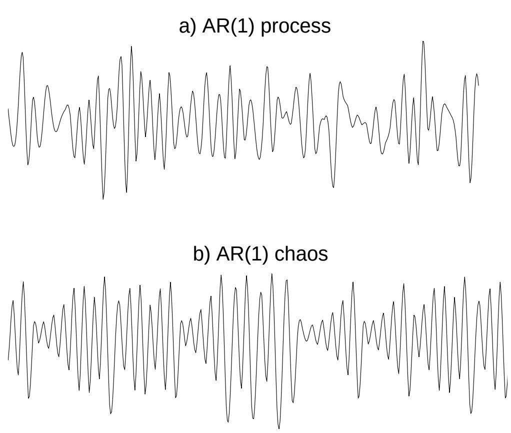

Figure 3: Comparison of interpolated signals from a) AR(1) process and b) AR(1) Chaotic process, both with $a = -0.6$.

5. CONCLUSIONS

In this work we have shown a chaotic alternative to AR(1) models, and developed parameter and signal estimators. The ML estimator has been considered, but its huge computational cost and inconsistency make it useless. Therefore, we have divided the model estimation problem in two stages: parameter and signal estimation. In the parameter estimation stage we have compared an estimator based on the autocorrelation, a gradient descent technique, and a Hard-Censoring LS (HCLS) estimator. In the signal estimation stage, the parameter and itinerary obtained previously are used to calculate the ML estimate of the signal. To avoid numerical instability we estimate the last point of the sequence, and iterate backwards. The combination of the HCLS and ML estimators has a low computational cost and shows good performance. Further lines of research include developing estimators for other maps and searching for chaotic AR(p) models.

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