A FOUR-PARAMETER QUADRATIC DISTRIBUTION

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ABSTRACT

Quadratic distributions such as time-frequency distributions and ambiguity functions have many useful applications. In some cases it is desirable to have a quadratic distribution of more than two variables. Using the technique of applying operators to variables, general quadratic distributions of more than two variables can be developed. We use this technique to develop a four-parameter quadratic distribution that includes variables of time, frequency, lag, and doppler. A general distribution is first developed and some of the mathematical properties are discussed. The distribution is then applied to the improvement of an adaptive time-frequency distribution. An example signal is shown to evaluate the performance of the technique.

1. FOUR PARAMETER DISTRIBUTION

Quadratic distributions have a variety of applications to fields such as time-frequency analysis, RADAR, and analysis of biological signals. Most two-variable quadratic distributions involve the variables of time and frequency, or as in the case of the ambiguity function, lag and doppler. In some cases it would be useful to have a joint distribution of all four of these variables. O'Neil and Williams [1] have developed a quartic version of a time, frequency, lag, and doppler distribution. In order to circumvent the problem of excessive cross-terms inherent to a quartic distribution, we develop here a quadratic version of such a distribution.

In order to develop the four-parameter distribution, we will first show how the ambiguity function (AF) and the Wigner distribution (WD) [2] can be cast into a general quadratic form using operator notation. The concept of applying operators to the signal is central to development of the four-parameter distribution. We utilize two operators in this development, the time-shift operator and the frequency-shift operator. The time-shift operator when applied to a signal in the time domain is defined as

$$(\mathbf{T}_{t_0}s)(u) = s(u - t_0) \tag{1}$$

The frequency-shift operator is defined as

$$(\mathbf{F}_{f_0}s)(u) = e^{j2\pi f_0 u}s(u) \tag{2}$$

Now examine the AF.

$$(\mathbf{AF}s)(\tau, \theta) = \int s(t)s^*(t-\tau)e^{-j2\pi\theta t}dt \tag{3}$$

The AF takes a signal and cross-correlates it with a time and frequency shifted version of the signal. The general quadratic integral representation easily accommodates such a cross-correlation interpretation.

$$\int\int K(u_1, u_2)s(u_1)s^*(u_2)du_1du_2 \tag{4}$$

Looking at a quadratic distribution as a cross-correlation often adds valuable insight.

The AF can be cast into the form of (4). We can start with the conceptual cross-correlation, which is the signal correlated with a time and frequency shifted version of itself.

$$(\mathbf{AF}s)(\tau, \theta) = \langle s, \mathbf{F}_{\frac{\theta}{2}}\mathbf{T}_{\frac{\tau}{2}}s\rangle \tag{5}$$

$$= \int s(t)s^*(t-\tau)e^{-j2\pi\theta t}dt \tag{6}$$

From (6), we can easily get to the form of (4) as follows.

$$(\mathbf{AF}s)(\tau, \theta) = \int s(t)s^*(t-\tau)e^{-j2\pi\theta t}dt \tag{7}$$

$$= \int e^{-j2\pi\theta(u_2+\tau)}s(u_2+\tau)s^*(u_2)du_2 \tag{8}$$

$$= \int\int e^{-j2\pi\theta(u_2+\tau)}\delta(u_2-\tau-u_1)s(u_1)s^*(u_2)du_1du_2$$

Where the kernel $K = e^{-j2\pi\theta(u_2+\tau)}\delta(u_2-\tau-u_1)$. The Wigner distribution will also be an important aspect of the four-parameter distribution and we would like to also define it in terms of the time and frequency shift operators. The subtle difference with the WD is the inversion of the variable of integration.

$$WD(t,f) = \int s(t+\frac{\tau}{2})s^*(t-\frac{\tau}{2})e^{j2\pi f\tau}d\tau \tag{9}$$

The WD is a cross-correlation of the signal with a time-reversed, time-shifted, and frequency-modulated version of itself. To see this more clearly, let's make a change of variables $\frac{\tau}{2} = u - t$.

$$WD(t,f) = \int s(t+\frac{\tau}{2})s^*(t-\frac{\tau}{2})e^{j2\pi f\tau}d\tau \tag{10}$$

$$= e^{j4\pi ft}\int s(u)s^*(-u+2t)e^{-j4\pi fu}du \tag{11}$$

$$WD(t^{`},f) = e^{j2\pi ft^{`}}\int s(u)s^*(-u+t^{`})e^{-j4\pi fu}du$$

We cross-correlate the signal with a version that is reversed in time, shifted in time by an amount $t^{`}$, and modulated by the complex sinusoid $e^{-j4\pi fu}$. This makes intuitive sense if we think about the WD. By shifting the signal in time and reversing it, we are folding a section of the future back onto the past, which tells us how localized in time a signal is. Modulating by the sinusoid gives us information about the frequency content at that time. The WD can also be represented in the form of (4) ;

$$(\mathbf{WD}s)(\tau,\theta) = \int\int K(u_1,u_2)s(u_1)s^*(u_2)du_1du_2 \tag{12}$$

$$= \int\int e^{j2\pi f(t-2u_1)}\delta(u_2+u_1-t)s(u_1)s^*(u_2)du_1du_2$$
$$= \int e^{j2\pi ft}e^{-j4\pi fu_1}s(u_1)s^*(-u_1+t)du_1$$
$$= WD(t,f)$$

Notice the form of the kernel that we used to get to the WD. It was a δ function that essentially forced an integration along a line in the two dimensional $u_1 - u_2$ plane. Cohen's class of time-frequency distributions is a concise way of describing many time-frequency distributions in terms of convolving a kernel with the WD. We can obtain Cohen's class from the general quadratic integral form with a little bit of work as shown [3].

$$(\mathbf{P}s)(t,f) = \langle \mathbf{K_P}\mathbf{F}_{-f}\mathbf{T}_{-t}s, \mathbf{F}_{-f}\mathbf{T}_{-t}s\rangle \tag{13}$$

$$= \int\int K_P(u_2,u_1)s(u_1+t)s^*(u_2+t) \tag{14}$$
$$e^{-j2\pi f(u_1-u_2)}du_1du_2$$
$$= \int\int \Pi(u,\tau)s(t+u+\frac{\tau}{2})s^*(t+u-\frac{\tau}{2})$$
$$e^{-j2\pi f\tau}dud\tau$$
$$= \int\int \Phi(u-t,v-f)(\mathbf{W}s)(u,v)dudv$$

Where $\mathbf{W}s$ represents the Wigner distributions of the signal s. The motivation for the four-parameter distribution is to combine the operator form of the AF (5) with the operator form of Cohen's class of time-frequency distributions (13). We will retain the generality of the quadratic integral form by including a yet unspecified kernel in the description. The end goal is to develop a time-frequency distribution that has variables of time, frequency, time lag, and frequency lag.

We start by simply combining the forms of (5) and (13);

$$(\mathbf{P}s)(t,f,\tau,\theta) = \langle \mathbf{K}_p\mathbf{F}_{\frac{-\theta}{2}}\mathbf{F}_{\frac{-\tau}{2}}\mathbf{F}_{\frac{-t}{2}}\mathbf{F}_{\frac{-f}{2}}s, \mathbf{F}_{\frac{\theta}{2}}\mathbf{F}_{\frac{\tau}{2}}\mathbf{F}_{\frac{-t}{2}}\mathbf{F}_{\frac{-f}{2}}s\rangle$$

$$= \int\int K_p(u_1,u_2)s(u_1+t+\frac{\tau}{2})s^*(u_2+t-\frac{\tau}{2})$$
$$e^{-j\pi\theta(u_1+u_2)}e^{-j2\pi f(u_1-u_2+\tau)}du_1du_2 \tag{15}$$

We now have a very general description of a t, f, τ, θ representation. We need to impose some further structure on the form of the kernel in order to gain some insight from such a general description. If we think of the signal in two-dimensional time-frequency space, we are trying to obtain the AF of a specific region that is bandlimited in both time and frequency. We could vary the kernel with time and frequency to excise a particular $t - f$ region. A better alternative is to make the kernel independent of t, f by making it a have low-pass properties in time and frequency and shifting the desired $t-f$ portion of the signal under the kernel using operators. Since (15) implements the desired $t - f$ shift, we follow the latter approach. We next discuss a few basic properties of the distribution.

2. PROPERTIES

Reduction to AF Looking at the distribution for only non-zero lag variables,

$$\int\int K_p(u_1,u_2)s(u_1+\frac{\tau}{2})s^*(u_2-\frac{\tau}{2})e^{-j\pi\theta(u_1+u_2)}du_1du_2 \tag{16}$$

If we select the kernel to be a delta function $\delta(u_1 - u_2)$, we obtain the AF

$$\int\int \delta(u_1 - u_2)s(u_1 + \frac{\tau}{2})s^*(u_2 - \frac{\tau}{2})e^{-j\pi\theta(u_1+u_2)}du_1du_2 \tag{17}$$

$$= \int s(u_1 + \frac{\tau}{2})s^*(u_1 - \frac{\tau}{2})e^{-j2\pi\theta u_1}du_1$$

$$= AF(\theta, \tau)$$

Reduction to WD There is an similar relationship for the case where we place zeros in the lag variables.

$$P(t, f, 0, 0) = \int\int K_p(u_1, u_2)s(u_1 + t)s^*(u_2 + t)$$
$$e^{-j2\pi f(u_1-u_2+\tau)}du_1du_2 \tag{18}$$

If we now select the kernel to be the delta function $\delta(u_1 + u_2)$, we get the following result.

$$= \int\int \delta(u_1 + u_2)s(u_1 + t)s^*(u_2 + t)$$
$$e^{-j2\pi f(u_1-u_2+\tau)}du_1du_2$$
$$= \int s(t + u_1)s^*(t - u_1)e^{-j2\pi f2u_1}du_1$$
$$= \int s(t + \frac{v}{2})s^*(t - \frac{v}{2})e^{-j2\pi fv}dv$$
$$= WD(t, f)$$

$T - F$ **Shift Property** If the signal is shifted in time and frequency, there is a corresponding shift in the distribution. If we take the signal $s(t)$ and shift it in time and frequency so that we have $s(t) \mapsto s(t - t_0)e^{j2\pi f_0 t}$, we get the following.

$$= \int\int K_P(u_1, u_2)s(u_1 + t - t_0 + \frac{\tau}{2})s^*(u_2 + t - t_0 - \frac{\tau}{2})$$
$$e^{-j\pi\theta(u_1+u_2)}e^{-j2\pi f(u_1-u_2+f_0)}du_1du_2$$
$$= P(t - t_0, f - f_0, \tau, \theta)$$

3. APPLICATION TO ADAPTIVE TFD

Time-frequency distributions that utilize data-adaptive kernels have proven to be useful on a wide variety of signals where minimal information about the signal is known a priori. Jones and Baraniuk [4] have developed an approach that adaptively develops a kernel as the solution to an optimization problem. It starts by restricting possible kernels to be of the form

$$\Phi(\theta, \tau) = exp(-\frac{\theta^2 + \tau^2}{2\sigma^2(\psi)}) \tag{19}$$

Where the function $\sigma(\psi)$ controls the spread of the Gaussian shaped kernel. The problem is more easily solved by a change to polar coordinates, which can be accomplished by the change of variables $r^2 = \theta^2 + \tau^2$. This gives the polar form of the kernel

$$\Phi(r, \psi) = exp(-\frac{-r^2}{2\sigma^2(\psi)}) \tag{20}$$

Given the form for the kernel, the problem is to select the kernel that is the solution to the following constrained optimization problem.

$$max_\Phi \int_0^{2\pi}\int_0^\infty |\Phi(r, \psi)|^2 drd\psi \tag{21}$$

Subject to the following constraint

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^\infty |\Phi(r, \psi)|^2 rdrd\psi = \frac{1}{4\pi^2}\int_0^{2\pi} \sigma^2(\psi)d\psi < \alpha \tag{22}$$

The goal is to find a kernel that encompasses as much of the signals AF plane energy as possible while satisfying the constraint that it have the form of (20) and its volume be limited to (22). Once this problem is solved, the TFD is easily computed by as the two-dimensional fourier transform.

$$(\mathbf{P}s)(t, f) = \int\int \Phi(\theta, \tau)AF((\theta, \tau)e^{-j2pi\theta t}e^{-j2pi\tau f}d\tau d\theta \tag{23}$$

Baraniuk and Jones have also proposed a modification to this procedure for signals with a high degree of nonstationarity [5]. If the AF in the above procedure is replaced by the short-time AF (STAF), we obtain an optimal kernel for each time point

$$AF(t; \theta, \tau) =$$
$$\int\int s^*(u - \frac{\tau}{2})w^*(u - t - \frac{\tau}{2}s(u + \frac{\tau}{2})w(u - t + \frac{\tau}{2})e^{j\theta u}du \tag{24}$$

The algorithm, dubbed adaptive optimal kernel (AOK) then uses each computed kernel to obtain a slice of the desired TFD.

$$\mathbf{P}_{aok}(t,f) = \int\int A(t;\theta,\tau)\Phi_{opt}(t;\theta,\tau)e^{-j2\pi\theta t - j2\pi f\tau}d\theta d\tau \quad (25)$$

The four-parameter distribution can be combined with the optimal- kernel procedure to obtain an algorithm that is sensitive to changes of the signal in time (like AOK) but also to changes in frequency.

The four-parameter distribution replaces the AF in the optimal -kernel procedure and an optimal kernel is computed. This kernel is then used to obtain a single (t, f) point of the TFD. The following test signal was used to illustrate the potential benefits of such a procedure.

$$x(t) = h_0(t)e^{-j.5\pi t} + h_1(t)e^{j.4\pi t} + h_1(t)e^{j.7\pi t} \quad (26)$$

where $h_0(t) = e^{\frac{t^2}{16}}$ and $h_1(t) = e^{\frac{t^2}{1024}}$

The next two figures show The TFD of the signal using the AOK procedure compared to the TFD using the four-parameter optimal-kernel procedure. The AOK procedure has problems with such a multicomponent signal because all of the components overlap in time. The four-parameter version of AOK is able to exploit the frequency separability of the components to produce a representation with good auto-term resolution yet with few interfering cross-terms.

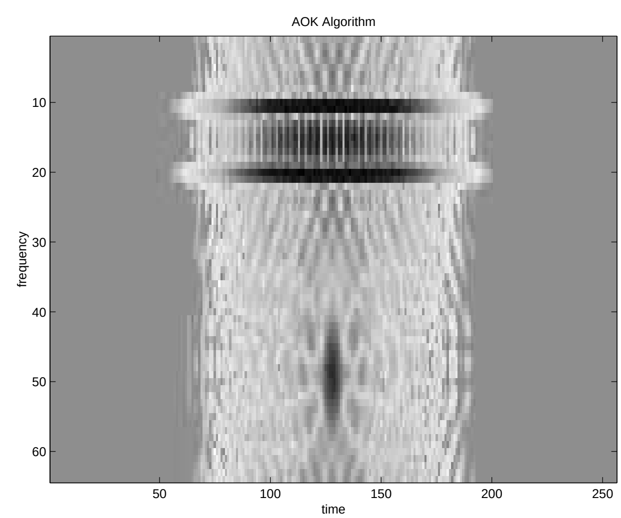

Fig. 1. AOK Representation of Test Signal

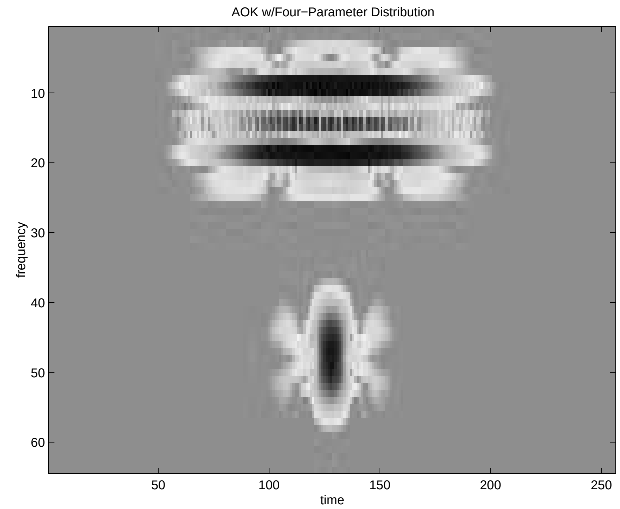

Fig. 2. Four-Parameter/AOK of Test Signal

4. REFERENCES

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