A MULTI-HYPOTHESIS GLRT APPROACH TO THE COMBINED SOURCE DETECTION AND DIRECTION OF ARRIVAL ESTIMATION PROBLEM

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ABSTRACT

The problem of detecting the number of uncorrelated narrowband signals received by an array of sensors when the direction of arrival (DOA) and power level of each source is unknown is investigated. A multi-hypothesis generalized likelihood ratio test (GLRT) approach is used, resulting in a procedure which maximizes the likelihood function with respect to the number of signals and their DOAs and powers. A tuning mechanism for controlling the trade-off between the probability of correct detection and the probability of false alarm is obtained by imposing a constraint on the minimum allowable value for the power level estimate. A sequential search over the number of sources is used for a computationally feasible solution. Performance comparisons are made to the Minimum Description Length (MDL) and Minimum Variance Distortionless Response (MVDR) signal detection approaches.

1. INTRODUCTION

The problem of detecting and localizing uncorrelated narrowband sources using an array of sensors is an important problem in many fields including radar, sonar, wireless communications, and medical imaging. The problem can be decomposed into a detection process where the number of sources is estimated, followed by an estimation process where the source parameters, (direction of arrival (DOA) and power level), are estimated. Unstructured model order estimation techniques such as Minimum Description Length (MDL) [1] may be used for the detection phase, and a direction finding technique such as maximum likelihood (ML), MUSIC, or ESPRIT (to name a few) may be used for the estimation phase. Alternately, a simultaneous detection and estimation procedure can be used, such as structured MDL [2] or Minimum Variance Distortionless Response (MVDR) [3].

The unstructured MDL detector analyzes eigenvalues of the sample covariance matrix for the "best" grouping of signal and noise eigenvalues. It is straightforward to implement, but does not exploit the physical and statistical structure of the data. MDL is known for its conservative estimates and has no mechanism to adjust the probability of false alarm. Therefore, a penalty is paid in detection sensitivity. The structured MDL detector operates on the same principle but makes more use of the data model. It is formulated for the case of unknown noise power and possibly correlated signals. It requires a maximum likelihood estimation of the source DOAs for each model order, which can be a considerable computational burden for large model orders. It also has no means for controlling false alarms and tends to underestimate the number of sources when the SNR is low. The MVDR detector computes a spatial spectrum, and estimates the number of signals from the number of peaks exceeding a threshold, with their DOAs given by the location of the peaks. It is easy to implement, and the detection threshold can be adjusted to trade-off false alarms and detection sensitivity. The MVDR procedure has difficulty detecting closely spaced targets.

In this paper, we take a generalized likelihood ratio test (GLRT) approach to the detection problem. For each hypothesized model order, we find the parameter estimates which maximize the likelihood function, and choose the hypothesis which has the largest likelihood. This results in a procedure which maximizes the likelihood function with respect to the number of signals and their parameters. A constraint is imposed on the minimum allowable power level estimate to obtain a tuning mechanism for controlling the trade-off between the probability of correct detection and the probability of false alarm. A sequential search over the number of sources is used for a computationally feasible solution. The technique is primarily a detection technique which provides parameter estimates as part of the detection process. The parameter estimates can always be further refined by an estimation process. Performance comparisons are made to the MVDR and unstructured MDL signal detection approaches.

2. MATHEMATICAL MODEL

Complex scalar data $z_{n,\kappa}$ is observed at each sensor n for N sensors at observation time κ for K observation times. Element n of Nx1 vector $\vec{z}_\kappa$ is $z_{n,\kappa}$. The observation model for $\vec{z}_\kappa$ is

$$\vec{z}_\kappa = \mathbf{V}^{(L)}\vec{s}_\kappa + \vec{w}_\kappa \qquad \kappa = 1, \ldots, K \tag{1}$$

where $\vec{w}_\kappa$ is a Nx1 vector of additive noise samples, $\vec{s}_\kappa$ is a Lx1 vector of source signal samples. $\mathbf{V}^{(L)}$ is a NxL matrix whose columns are the array response vectors for each of the L sources,

$$\mathbf{V}^{(L)} = [\vec{v}(u_1) \ \cdots \ \vec{v}(u_L)], \tag{2}$$

where u_l is the DOA of the lth signal. All signal samples $s_{\kappa,l}$ and noise samples $w_{\kappa,n}$ are uncorrelated complex zero mean Gaussian random variables. The LxL signal covariance matrix $\mathbf{S}_s^{(L)}$ is the diagonal matrix

$$\mathbf{S}_s^{(L)} = \mathrm{E}[\vec{s}_\kappa \vec{s}_\kappa^{\,\mathsf{H}}] = \mathrm{Diag}(\sigma_l^2) \quad l = 1, \ldots, L \tag{3}$$

where σ_l^2 is the power of signal l. The NxN noise covariance matrix is given by $\mathrm{E}[\vec{w}_\kappa \vec{w}_\kappa^{\,\mathsf{H}}] = \sigma_n^2\,\mathbf{I}$, where σ_n^2 is the noise power. It is assumed to be known and the same for all sensors. The NxN observation data covariance matrix $\mathbf{S}_z^{(L)}$ has the form

$$\mathbf{S}_z^{(L)} = \mathrm{E}[\vec{z}_\kappa \vec{z}_\kappa^{\,\mathsf{H}}] = \mathbf{V}^{(L)}\mathbf{S}_s^{(L)}\mathbf{V}^{(L)\mathsf{H}} + \sigma_n^2\,\mathbf{I}. \tag{4}$$

The number of signals L, the signal DOA's $u_1, \ldots, u_L$, and the signal powers $\sigma_1^2, \ldots, \sigma_L^2$ are assumed constant but unknown over the entire observation time interval.

Let $\mathbf{z}$ denote the collection of K observation vectors $\vec{z}_\kappa$. The probability density function (PDF) of all observed data is given by

$$\mathrm{p}_L(\mathbf{z}) = \frac{1}{\pi^{KN}|\mathbf{S}_z^{(L)}|^K} \exp\left(-\sum_{\kappa=1}^{K} \vec{z}_\kappa^{\,\mathsf{H}} \mathbf{S}_z^{(L)^{-1}} \vec{z}_\kappa\right). \quad (5)$$

We will find it convenient to normalize with respect to the noise only PDF $\mathrm{p}_0(\mathbf{z})$ obtained when no signals are present ($L = 0$). Using the matrix inversion lemma, the L signal log likelihood ratio (LLR) $\lambda_L(\mathbf{z}) = \ln\left(\mathrm{p}_L(\mathbf{z})/\mathrm{p}_0(\mathbf{z})\right)$ has the form

$$\lambda_L(\mathbf{z}) = \frac{1}{\sigma_n^2}\sum_{\kappa=1}^{K} \vec{z}_\kappa^{\,\mathsf{H}} \mathbf{A}^{(L)} \vec{z}_\kappa - K \ln\left|\sigma_n^{-2}\mathbf{S}_z^{(L)}\right|, \quad (6)$$

$$\text{where} \quad \mathbf{A}_L = \mathbf{V}^{(L)}\left(\mathbf{V}^{(L)\mathsf{H}}\mathbf{V}^{(L)} + \sigma_n^2 \mathbf{S}_s^{(L)^{-1}}\right)^{-1}\mathbf{V}^{(L)\mathsf{H}}. \quad (7)$$

3. MULTIHYPOTHESIS GLRT APPROACH

The problem can be formulated as a multi-hypothesis detection problem, where the hypotheses correspond to the number of signals. If we could specify the DOAs and power levels of the L sources for each hypothesis, we could compute $\lambda_L(\mathbf{z})$ for $L = 0, \ldots, L_{\max}$, and choose the hypothesis corresponding to the largest value. However, the DOAs and power levels are unknown, thus there are $2L$ unknown parameters for each hypothesis. We can take a generalized likelihood ratio test (GLRT) approach and find the parameter estimates which maximize the likelihood function for each hypothesized number of sources, and choose the largest. The generalized log likelihood ratio (GLLR) then becomes

$$\hat{\lambda}_L(\mathbf{z}) = \max_{\substack{u_1 \cdots u_L, \\ \sigma_1^2 \cdots \sigma_L^2}} \frac{1}{\sigma_n^2}\sum_{\kappa=1}^{K} \vec{z}_\kappa^{\,\mathsf{H}} \mathbf{A}^{(L)} \vec{z}_\kappa - K \ln\left|\sigma_n^{-2}\mathbf{S}_z^{(L)}\right|. \quad (8)$$

This is the basic approach studied here. The main drawbacks of the technique as stated are that it becomes increasingly difficult to estimate the unknown parameters as the number of hypothesized signals increases, and there is no mechanism for trading off the probability of correctly detecting the number of signals versus the probability of detecting too few or too many signals.

The complexity issue can be handled by computing the GLLR for increasing L sequentially using the parameter estimates from the previous stage. First note that the GLLR $\hat{\lambda}_0(\mathbf{z})$ requires no optimization and is equal to zero. We start by finding the GLLR $\hat{\lambda}_1(\mathbf{z})$ assuming one signal present. When $L = 1$, (8) reduces to

$$\hat{\lambda}_1(\mathbf{z}) = \max_{u_1, \sigma_1^2} \frac{K}{\sigma_n^2} \frac{\vec{v}(u_1)^{\mathsf{H}} \hat{\mathbf{R}}_z \vec{v}(u_1)}{N + \sigma_n^2/\sigma_1^2} - K\left(1 + N\frac{\sigma_1^2}{\sigma_n^2}\right), \quad (9)$$

where $\hat{\mathbf{R}}_z$ is the sample covariance matrix $\hat{\mathbf{R}}_z = \frac{1}{K}\sum_{\kappa=1}^{K} \vec{z}_\kappa \vec{z}_\kappa^{\,\mathsf{H}}$. The optimum power estimate as a function of DOA is given by

$$\hat{\sigma}_1^2(u_1) = \max\left(\sigma_{nom}^2, \frac{\vec{v}(u_1)^{\mathsf{H}} \hat{\mathbf{R}}_z \vec{v}(u_1)}{N^2} - \frac{\sigma_n^2}{N}\right). \quad (10)$$

The power estimate is constrained to be no less than some nominal power level σ_{nom}^2. The standard ML estimate uses $\sigma_{nom}^2 = 0$, however this results in excessive false alarms in the GLRT. The nominal power level serves as a tuning devise to control the probability of false alarm, at the expense of a reduction in detection sensitivity. The power estimate in (10) is substituted back into (9), and the result maximized with respect to the DOA. If $\hat{\lambda}_1(\mathbf{z}) < \hat{\lambda}_0(\mathbf{z}) = 0$, stop the search. The final solution is no signals present ($L = 0$). If $\hat{\lambda}_1(\mathbf{z}) > 0$, compute $\hat{\lambda}_2(\mathbf{z})$. The DOA and power from the $L = 1$ solution are fixed initially. Again, a closed form expression for $\hat{\sigma}_2^2$ is obtained subject to a minimum power level constraint, and a search is performed over the DOA u_2. The previous parameter estimates $\hat{u}_1$ and $\hat{\sigma}_1^2$ are then refined in an iterative manner to find $\hat{\lambda}_2(\mathbf{z})$. If $\hat{\lambda}_2(\mathbf{z}) < \hat{\lambda}_1(\mathbf{z})$, stop the search. The final solution is $L = 1$. If not, continue the search. This procedure is repeated for as many model orders as necessary until the process is stopped.

Investigation of the LLR $\lambda_L(\mathbf{z})$ in (6) reveals that the effective role of the determinant $|\sigma_n^{-2}\mathbf{S}_z^{(L)}|$ is a penalty function for the number of signals. For a fixed L, the determinant is largest when the signals are spatially orthogonal ($\mathbf{V}^{(L)\mathsf{H}}\mathbf{V}^{(L)} = N\,\mathbf{I}$). The orthogonal signals determinant is the product $|\sigma_n^{-2}\mathbf{S}_z^{(L)}| = \prod_{l=1}^{L}\left(1 + N\sigma_l^2/\sigma_n^2\right)$. The determinant decreases as source DOAs become closer, and the procedure tends to favor estimating two signals at the same location, rather than a single signal with the same total power. This problem can be alleviated by modifying the determinant term to always have the orthogonal signals form regardless of the DOA estimates. The penalty function is now more indicative of the physical number of signals rather than the effective number. The modified GLLR becomes

$$\hat{\lambda}_L(\mathbf{z}) = \frac{1}{\sigma_n^2}\sum_{\kappa=1}^{K} \vec{z}_\kappa^{\,\mathsf{H}} \hat{\mathbf{A}}^{(L)} \vec{z}_\kappa - K\sum_{l=1}^{L}\ln\left(1 + \frac{\hat{\sigma}_l^2}{\sigma_n^2}\right) \quad (11)$$

where $\hat{\mathbf{A}}^{(L)}$ is given by (7) with the estimated DOAs $\hat{u}_1, \ldots, \hat{u}_L$ and powers $\hat{\sigma}_1^2, \ldots, \hat{\sigma}_L^2$. This modification constitutes the final implementation.

4. SIMULATION RESULTS

For purposes of simulation results, a standard uniform linear array is assumed with number of observation times K equal 20 and number of sensors N equal 15.

4.1. No Signals Present Scenario

The first scenario is a no signals present scenario. Figures 1, 2, and 3 show the one signal LLR with estimated signal power vs. u_1 for three different nominal power constraints. $\hat{\lambda}_1(\mathbf{z})$ is the value of the LLR at the highest peak. If this value is larger than zero, then the location of the peak is the estimate of u_1. The desired performance for the no signals present scenario is $\hat{\lambda}_1(\mathbf{z}) < 0$, so that the test is stopped and no signals are declared.

In Figs. 1, 2, and 3, σ_{nom}^2 corresponding to $-\infty$ dB, -6 dB, and +6 dB ASNR (ASNR = $N\sigma_{nom}^2/\sigma_n^2$) respectively are used. Fig. 1 is the unmodified ML signal power case. Performance in Fig. 1 appears unusual (all LLR$\geq$0), but is normal. No information is given as to expected signal power except that it is positive, thus every noise spike is interpreted as a signal. Performance improves as the nominal signal power is increased. In Fig. 2, there are several noise spikes which exceed the threshold, while in Fig.

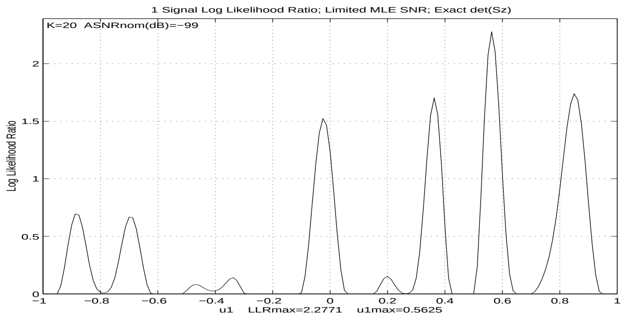

Fig. 1. Single signal log likelihood ratio with estimated power vs. DOA No signals present. Nominal ASNR set to zero (-∞ dB).

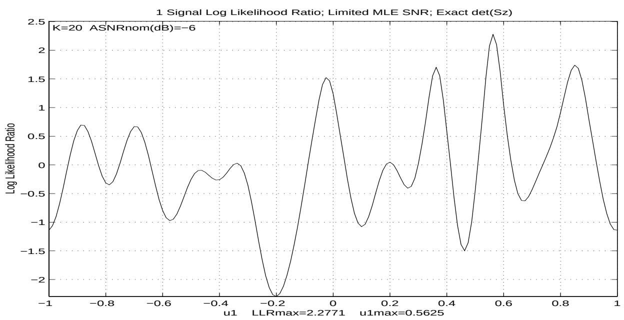

Fig. 2. Single signal log likelihood ratio with estimated power vs. DOA No signals present. Nominal ASNR set to -6 dB.

3, the nominal power is set high enough to discriminate against noise spikes.

The probability of false alarm for 1000 trials is shown in Table 1 for various nominal ASNR's. The purpose of Table 1 is to demonstrate that P_f can be controlled by selection of nominal ASNR. Higher nominal ASNR lowers P_f. The trade-off is that it also lowers probability of detection (P_d) for a fixed actual ASNR. Based on the results in Table 1, a nominal ASNR of 6 dB is used for all remaining simulations.

4.2. One Signal Present Scenario

In this scenario, one signal present is simulated for the purposes of testing the orthogonal determinant modification against the exact determinant solution. Desired performance for this scenario is $\hat{\lambda}_1(\mathbf{z}) > 0$ and $\hat{\lambda}_2(\mathbf{z}) < \hat{\lambda}_1(\mathbf{z})$. The probability of correct detection is shown in Table 2 for both procedures as a function of source ASNR. Both procedures have difficulty detecting the signal at low ASNR. At high ASNR, the orthogonal determinant procedure correctly finds only one signal near the correct DOA. The exact determinant procedure has difficulties at high ASNR due to detecting two coincident signals.

Nominal ASNR (dB)	0	3	6	9
P_f	0.801	0.219	0.004	0.000

Table 1. False alarm performance for 1000 trials.

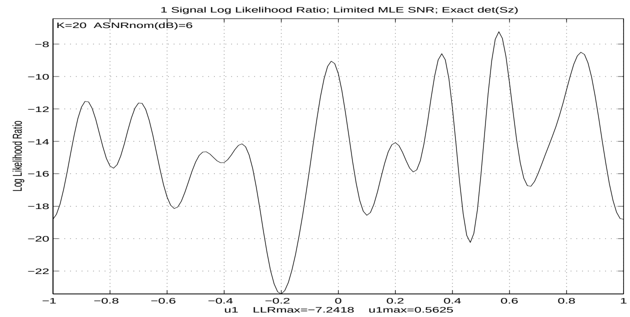

Fig. 3. Single signal log likelihood ratio with estimated power vs. DOA No signals present. Nominal ASNR set to 6 dB.

Sig. ASNR (dB)	-6	0	6	12	18
Exact Det. P_d	0.021	0.501	0.992	0.451	0.053
Orth. Det. P_d	0.021	0.501	0.997	0.992	0.988

Table 2. Detection performance for a single signal for 1000 trials.

4.3. Three Signals Present Scenario

A walk-through of the final implementation is given. Three signals are present. Two are closely spaced at DOA's $u_1 = -0.333$ and $u_2 = -0.28$ with 15 and 9 dB ASNR respectively. The third widely separated signal has DOA $u_3 = 0.4$ with 3 dB ASNR.

This test is successful. The correct number of signals is found, and accurate estimates of the signal DOA's are found. As desired, the LLR increases over the one, two, and three signals search and then decreases at the four signals search, so that the test is stopped and three signals are declared. The single source LLR is shown vs. DOA u_1 in Fig. 4. There is a single peak near the stronger of the two closely spaced signals with some bias toward the weaker signal. A peak is also seen at the third widely separated weaker signal. The two source LLR with the first source DOA and power level held fixed is shown vs. DOA u_2 in Fig. 5. The LLR imposes a null at DOA u_1. The second closely spaced signal is now revealed due to this null and is the peak. The three source LLR with the first two source's parameters held fixed is shown vs. DOA u_3 in Fig. 6. Desired nulls are seen at DOA's u_1 and u_2. The peak is at the widely separated signal DOA. The four source LLR with the first three source's parameters held fixed is shown vs. DOA u_4 in Fig. 7. The four signals present LLR decreases over the three signals present LLR as desired. The key to this performance is the desired nulls at DOA's u_1, u_2, and u_3. The signal energy has been suppressed with only noise energy left. The DOA estimates are sufficiently good to allow the detection process to perform as desired. More accurate estimates can always be found by a subsequent estimation process.

5. PERFORMANCE COMPARISONS

Signal detection performance is compared among three approaches: 1) the presented GLLR approach 2) unstructered MDL, and 3) MVDR. Probability of correct number of signals is empirically estimated on simulated data from 100 trials. The MVDR detection threshold is set to have equivalent probability of false alarm as the GLLR procedure. The MDL procedure has no mechanism to control false alarms, and appears to almost never falsely declare a signal present.

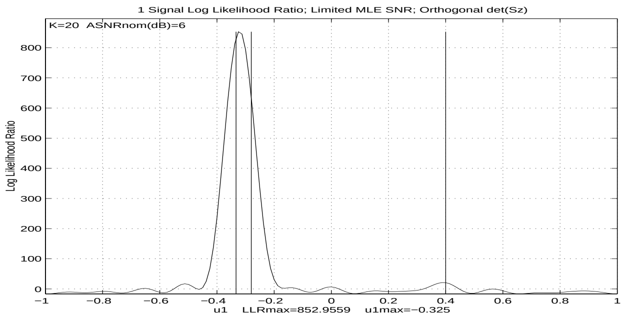

Fig. 4. Three Signals Present. One signal LLR vs. u_1.

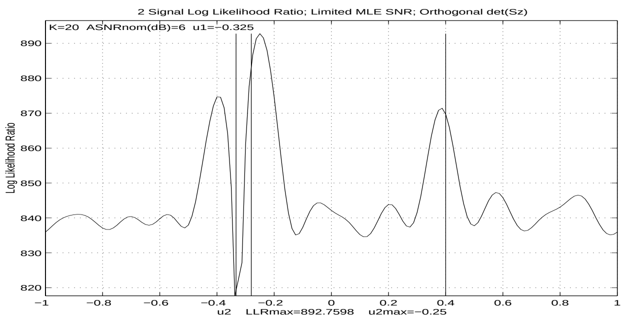

Fig. 5. Three signals present. Two signal LLR vs. u_2.

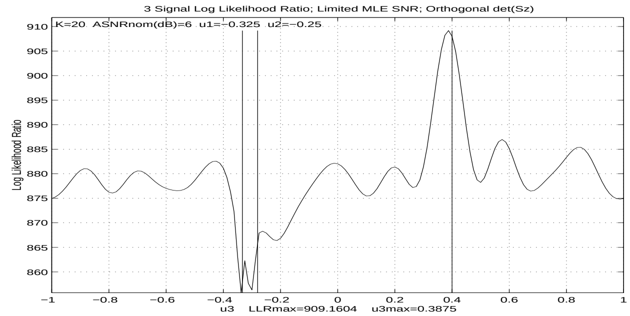

Fig. 6. Three signals present. Three signal LLR vs. u_3.

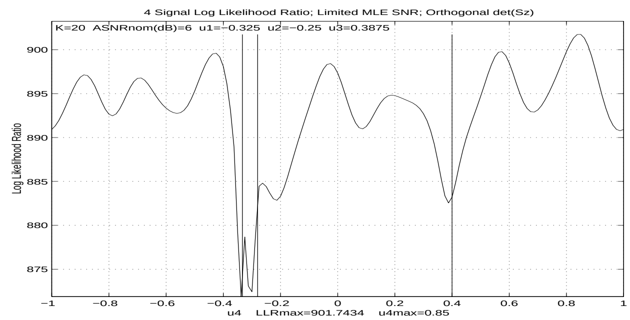

Fig. 7. Three signals present. Four signal LLR vs. u_4.

The first case is a one signal present scenario. The probability of correct detection is shown in Fig. 8 as a function of the source ASNR. At low ASNR, GLLR and MVDR are identical as expected. GLLR and MVDR significantly outperform MDL (5 dB better) in this region. MDL does not exploit the physical and statistical structure of the one signal case as does GLLR and MVDR. At high SNR, MDL approaches perfect detection, while GLLR and MVDR are slightly lower. This is the trade-off for improved performance at low SNR.

The second case is a two signals present scenario. The ASNR of one signal is held constant at 9 dB ASNR. The ASNR of the second signal is varying. The same closely spaced signal DOA's are used as in Fig. 4. Fig. 9 shows probability of correct detection as a function of the second signal ASNR. GLLR outperforms both MDL and MVDR. MVDR is not able to resolve the signals and always underestimates the number of sources. MDL clearly cannot perform on this scenario. On widely spaced signals, detection of each signal is an approximately independent event and similar performance to Fig. 8 is obtained.

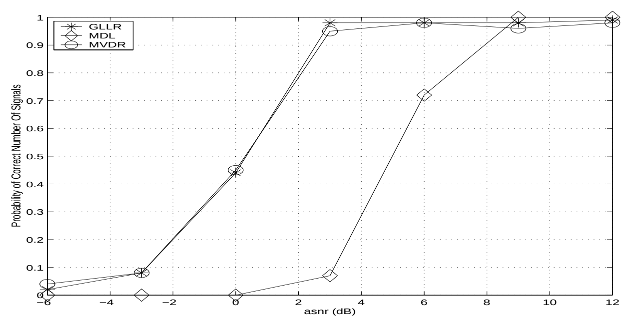

Fig. 8. One signal detection performance

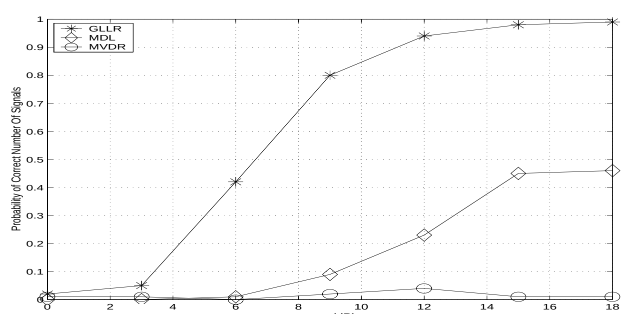

Fig. 9. Two signal detection performance.

6. REFERENCES

[1] M. Wax and T. Kailath, "Detection of Signals by Information Theoretic Criteria," *IEEE Trans. ASSP*, vol. 33, no. 2, pp. 387-392, April 1985.

[2] M. Wax, "Detection and Localization of Multiple Sources via the Stochastic Signals Model," *IEEE Trans. Sig. Proc.*, vol. 39, no. 11, pp. 2450-2456, Nov. 1991.

[3] J. Capon, "High-Resolution Frequency-Wavenumber Spectrum Analysis", *Proc. IEEE*, vol. 57, pp. 1408-1418, Aug. 1969.