SEQUENTIAL FORWARD SAMPLE SELECTION IN ARRAY-BASED IMAGE FORMATION

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ABSTRACT

In some types of imaging, the signal is strictly limited in one domain while sampling takes places in another. If sampling is done in a rectangular array pattern at sub-Nyquist density, the array must be dithered to sample the image at the Nyquist density in each dimension. However, the Nyquist density oversamples the image due to the nonrectangular support in the transform domain. We present an efficient forward selection algorithm for optimizing the dithering pattern so that the image can be reconstructed as reliably as possible from a periodic nonuniform set of samples, which can be obtained from a dithered rectangular-grid array. Our examples show that this new algorithm makes selective sampling possible in a real-time image acquisition setting for MR spectroscopic imaging.

1. INTRODUCTION

In some types of imaging, the signal is strictly limited in one domain while sampling takes places in another. This is the case in passive millimeter-wave imaging, in which the finite aperture produces strictly bandlimited images with a circular frequency support while sampling takes places in the spatial domain. Also, in MRI, sampling occurs in the spatial-frequency domain, but the spatial-domain image sometimes has a limited region of support (ROS). Either for convenience or due to hardware limitations, sampling is often done in a rectangular array pattern at sub-Nyquist density [1–3]. The array must be dithered to sample the image at the Nyquist density in each dimension. However, the Nyquist density oversamples the image due to the nonrectangular support in the transform domain [4].

In previous work, we developed a sequential backward selection method for optimizing the dithering pattern [5]. Taking into account the transform support of the image, we sequentially eliminated the least informative array recursively until the minimal number of arrays remain. This approach has two drawbacks. First, the method is too computationally intensive for some applications. Second, in some real-time imaging applications — MRI, for instance — we desire a method that can select rather than eliminate one array at a time so that the imaging process can run in parallel with the selection process. No existing technique allows one to sequentially select rectangular arrays.

In this paper, we develop an efficient sequential forward selection algorithm. The computational complexity is lowered considerably by the changed structure of the selection criterion that is used for forward selection. Furthermore, imaging can begin as soon as the first array is selected. As long as the selection algorithm is faster than the imaging process, no delay occurs due to sampling.

2. MATHEMATICS OF SAMPLING

2.1. Structure of the Observation Equation

We consider a sampling geometry in which the individual samples are laid out in a rectangular grid pattern as shown in Figure 1(a). The heavy dots represent the locations of the samples in the unshifted sampling array. The light dots represent other locations to which the sampling array can be shifted. If we allow nonuniform sampling we can reduce the average sampling density below the Nyquist density [4]. In particular, we can sample an image using a periodic replication of a nonperiodic sampling pattern to capture the information in a bandlimited image without aliasing. An example of this kind of sampling pattern is shown in Figure 1(b). In this example, each 3×3 block sampling pattern is periodically replicated over the image plane. This pattern results from shifting the array in Figure 1 according to each offset sample in one of the 3×3 blocks.

Without loss of generality, our derivation assumes

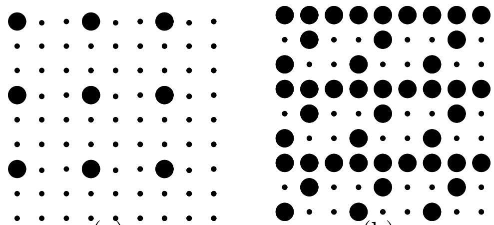

Figure 1: (a) Sensor array sampling pattern. (b) Periodic nonuniform sampling example.

that sampling is done in the frequency domain and the image is spatially limited. If we represent the frequency samples as a vector y, the unknown discretized spatial support by a vector x, and the mapping from the spatial samples to the frequency domain by F, the fully sampled frequency domain can be expressed as

$$y = Fx + u \tag{1}$$

where u is zero-mean, i.i.d. noise. If we sample y using a single position of the sampling array with offset indexed by i, we can represent this by

$$y_i = Q_i y \tag{2}$$

where Q_i downsamples the fully sampled frequency domain and orders the result into a vector. Then we can rewrite y in a rearranged form y_r as

$$\begin{aligned} y_r &= [y_1^H \; y_2^H \; \ldots \; y_n^H]^H \\ &= [Q_1^H \; Q_2^H \; \ldots \; Q_n^H]^H (Fx + u) \\ &= Q_r F x + u_r \end{aligned} \tag{3}$$

where u_r is the similarly rearranged version of u and $Q_r = [Q_1^H \; Q_2^H \; \ldots \; Q_n^H]^H$. If we choose a subset of k of the n shifted arrays, we obtain

$$\begin{aligned} \tilde{y}_r &= [y_{n_1}^H \; y_{n_2}^H \; \ldots \; y_{n_k}^H]^H \\ &= [Q_{n_1}^H \; Q_{n_2}^H \; \ldots \; Q_{n_k}^H]^H (Fx + u) \\ &= \tilde{Q}_r F^H y + \tilde{u}_r \end{aligned} \tag{4}$$

where $\tilde{u}_r$ is the corresponding subset of u_r and $\tilde{Q}_r$ is the corresponding subset of Q_r.

As long as the $\{Q_i\}$ are properly chosen and enough subsets are selected, the unknown samples x can be reconstructed from $\tilde{y}_r$ in a least-squares sense by

$$\hat{x} = (F^H \tilde{Q}_r^H \tilde{Q}_r F)^{-1} F^H \tilde{Q}_r^H \tilde{y}_r \tag{5}$$

2.2. Selection Criterion

Assuming that u has unit variance, the sum of squared errors (SSE) in the reconstructed image is given by

$$\phi(\tilde{Q}_r) = \mathrm{tr}\,(F^H \tilde{Q}_r^H \tilde{Q}_r F)^{-1} \tag{6}$$

Unfortunately, this criterion is unsuitable for a forward selection algorithm, since $\tilde{Q}_r F$ will have more columns than rows at the beginning of the selection process, making the criterion undefined. In a previous paper [6], we proposed the use of the following modified criterion:

$$\phi_m(\tilde{Q}_r) = \mathrm{tr}\,(\tilde{Q}_r F F^H \tilde{Q}_r^H)^{-1} \tag{7}$$

This criterion reflects the sum of squared errors due to noise in a minimum-norm least-squares solution. Because $F^H \tilde{Q}_r^H$ begins with more rows than columns, the criterion is defined initially.

As samples are selected, the number of columns in $F^H \tilde{Q}_r^H$ increases. Eventually, if we select more samples than unknowns, the number of columns will exceed the number of rows, forcing (7) to be undefined at that point (if not before). Therefore, this criterion cannot be used throughout the selection process either. Instead, we propose to use the following:

$$\phi_\varepsilon(\tilde{Q}_r) = \mathrm{tr}\,(\tilde{Q}_r F^H F \tilde{Q}_r^H + \varepsilon I)^{-1} \tag{8}$$

This criterion is always defined. We can show that when the number of selected samples is less than or equal to the number of unknowns

$$\phi_m(\tilde{Q}_r) = lim_{\varepsilon \to 0} \phi_\varepsilon(\tilde{Q}_r) \tag{9}$$

where $f(k)$ is a function of the number of selected samples and not of $\tilde{Q}_r F$ as long as $\tilde{Q}_r F$ is chosen not to be rank-deficient. When the number of selected samples is greater than or equal to the number of unknowns

$$\phi(\tilde{Q}_r) = lim_{\varepsilon \to 0} [\phi_\varepsilon(\tilde{Q}_r) - \frac{1}{\varepsilon} f(k)] \tag{10}$$

where $f(k)$ is a function of the number of selected samples and not of $\tilde{Q}_r F$ as long as $\tilde{Q}_r F$ is chosen not to be rank-deficient. This means that if we choose ε small, we can use (8) to obtain the approximate selection performance of (7) when fewer samples than unknowns have been chosen and the performance of (6) after that. The term $\frac{1}{\varepsilon} f(k)$ in (10) will have no effect on the selection, since it is not a function of the particular array being selected in each step.

3. BASIC OPTIMIZATION ALGORITHM

To simplify the computational method, we first transform the block matrix in (8) into a diagonal block matrix after some modulation.

$$\begin{aligned} &\tilde{Q}_r F F^H \tilde{Q}_r^H \\ &= [Q_{n_1}^H \, Q_{n_2}^H \cdots Q_{n_k}^H]^H F F^H [Q_{n_1} \, Q_{n_2} \cdots Q_{n_k}] \end{aligned}$$

$$=\begin{bmatrix} Q_{n_1}FF^HQ_{n_1}^H & Q_{n_1}FF^HQ_{n_2}^H & \cdots & Q_{n_1}FF^HQ_{n_k}^H \\ Q_{n_2}FF^HQ_{n_1}^H & Q_{n_2}FF^HQ_{n_2}^H & \cdots & Q_{n_2}FF^HQ_{n_k}^H \\ \vdots & \vdots & \cdots & \vdots \\ Q_{n_k}FF^HQ_{n_1}^H & Q_{n_k}FF^HQ_{n_2}^H & \cdots & Q_{n_k}FF^HQ_{n_k}^H \end{bmatrix} \tag{11}$$

$Q_iFF^HQ_j^H, i, j = 1, 2, \cdots, n$, are easily seen to be circulant matrices, since FF^H is circulant. Furthermore, $Q_iFF^HQ_i$ are the same for $i = 1, 2, \cdots, n$ because FF^H is a circulant matrix. Therefore, (11) is a conjugate symmetric matrix with circulant blocks (circulant-block-circulant blocks in the 2-D case).

Let F_d represent a Fourier matrix for a downsampled array corresponding to the dithered array. Then $F_d^HF_d = I$. $F_dQ_iFF^HQ_jF_d^H$ diagonalizes the circulant matrix $Q_iFF^HQ_j, i, j = 1, 2, \cdots, n$. Define B_{ij} and d_{ijg} such that

$$\begin{aligned} B_{ij} &= diag(d_{ij1}, d_{ij2}, \cdots, d_{ijm}) \\ &= F_dQ_iFF^HQ_jF_d^H \end{aligned} \tag{12}$$

Also, let F_D be a block-diagonal matrix formed by replicating F_d along the diagonal k times, Finally, define

$$(D_g + \varepsilon I)^{-1} = \begin{bmatrix} d_{11g} + \varepsilon & d_{12g} & \cdots & d_{1kg} \\ d_{21g} & d_{22g} + \varepsilon & \cdots & d_{2kg} \\ \vdots & \vdots & \cdots & \vdots \\ d_{k1g} & d_{k2g} & \cdots & d_{kkg} + \varepsilon \end{bmatrix}^{-1} \tag{13}$$

for $g = 1, 2, \cdots, m$. Then

$$\begin{aligned} &(\tilde{Q}_rFF^H\tilde{Q}_r^H + \varepsilon I)^{-1} \\ =&F_DF_D^H(\tilde{Q}_rFF^H\tilde{Q}_r^H + \varepsilon I)^{-1}F_DF_D^H \\ =&F_D(F_D\tilde{Q}_rFF^H\tilde{Q}_r^HF_D^H + \varepsilon I)^{-1}F_D^H \\ =&F_D \begin{bmatrix} B_{11} + \varepsilon I & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} + \varepsilon I & \cdots & B_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ B_{k1} & B_{k2} & \cdots & B_{kk} + \varepsilon I \end{bmatrix}^{-1} F_D^H \\ =&F_D \left[\tilde{Q}_r\, blockdiag(D_1 + \varepsilon I, \ldots, D_m + \varepsilon I)\, \tilde{Q}_r^H\right]^{-1} F_D^H \\ =&F_D\tilde{Q}_r\, blockdiag((D_1 + \varepsilon I)^{-1}, \ldots, (D_m + \varepsilon I)^{-1})\, \tilde{Q}_r^HF_D^H \end{aligned} \tag{14}$$

We calculate the vector d_{ij} as follows, using FFT's instead of matrix operations.

1. Take the FFT of the ROS image to get the impulse response h, which is the first column of FF^H. FF^H is a circulant matrix. Only the first column needs to be stored.

2. Shift h by j, then downsample the shifted h by Q_i to get the first column h_d of the circulant matrix $Q_iFF^HQ_j, i, j = 1, 2, \cdots, n$.

3. Take the FFT of h_d to get the diagonal vector of the diagonal matrix $F_dQ_iFF^HQ_jF_d^H$.

By using the property $\mathrm{tr}\,(ABC) = \mathrm{tr}\,(CAB)$ and the fact that $F_D^HF = I$ and $Q_r^HQ_r = I$, we have

$$\mathrm{tr}\,(\tilde{Q}_rFF^H\tilde{Q}_r^H + \varepsilon I)^{-1} == \sum_{i=1}^{m} \mathrm{tr}\,(D_i + \varepsilon)^{-1} \tag{15}$$

Using the method detailed in [6], we can derive an efficient method for comparing each array using the available $(D_i + \varepsilon)^{-1}$ matrices so that one array is selected in each step. Furthermore, we can efficiently update the $(D_i + \varepsilon)^{-1}$ matrices when new arrays are selected without having to perform a matrix inverse. This leads to a highly efficient selection algorithm.

4. EXPERIMENTS

Figure 2(a) shows the water image from a full 64×64 k-space (spatial-frequency-domain) 1H MR spectroscopic imaging (MRSI) data set (courtesy of the Center for Nuclear Imaging Research, University of Alabama at Birmingham). The spatial ROS identified by hand is shown in Figure 3(a). It contains 1561 possibly nonzero voxels. Using this ROS, the k-space data were selected using a forward selection algorithm that selects shifted uniform arrays using the criterion (8) with $\varepsilon = 10^{-3}$. We selected 1792 out of 4096 samples. The algorithm required only 26 seconds in Matlab on a Sun Ultra 1. The sampling pattern is shown in Figure 3(b). A randomly shifted set of uniform arrays also containing 1792 samples was selected for comparison purposes. This pattern is shown in Figure 3(c). As a further comparison, we attempted to construct by hand a sample distribution as evenly spaced in the 64×64 grid as possible with 1792 samples. The evenly spaced distribution is shown in Figure 3(d).

The reconstruction from the SFS-selected samples is shown in Figure 2(d). The reconstruction from the randomly selected sample set is shown in Figure 2(b). The reconstruction from optimal selection is slightly noisier than the image reconstructed using all the data, but it is reconstructed from only 44% of the data of the original. If random selection is used, the reconstruction tends to have much greater reconstruction error and artifacts, as illustrated in the figure. In this case, the MSE is about six times higher than the reconstruction from optimized sampling, if the full 64×64 data is considered to be ground truth. As it turns out, the system defined by the evenly spaced sample distribution is singular! Clearly, nearly uniform spacing is not a desirable choice.

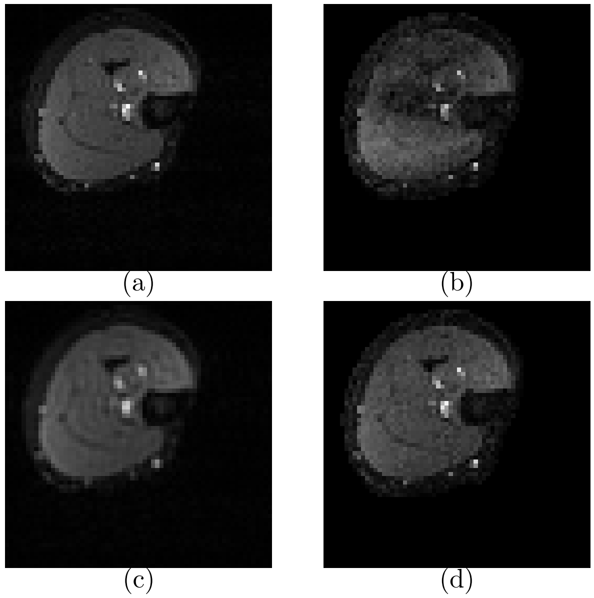

Figure 2: Water images: (a) from FFT of full data, (b) from random data, (c) from lowpass data and zeropadded FFT, (d) from optimized samples.

We also show an image reconstructed from lowpass sampling. Approximately the same number (1793) of low-frequency samples in a circle around DC were selected. The Fourier domain was then zeropadded and the image reconstructed from an FFT (Figure 2(c)). A close examination of the lowpass image shows that the reconstruction has about half the resolution of the optimized image. If the lowpass image is reconstructed by a least-squares technique with the ROS used as a constraint, the resulting system is nearly singular and the image is swamped by noise.

5. DISCUSSION

The optimized sampling scheme can reliably reduce the overall sampling requirements without any loss of resolution while also controlling noise amplification. In fact, our simulations show that if we reduce the number of samples in each shifted array and select a larger number of arrays to maintain the same overall number of samples, we can reduce the MSE by a further factor of two in the example in the experiment. This comes at the expense of longer computational time — about 6 minutes in this case. Most importantly for selective sampling in MRSI, the increased speed achieved by this algorithm makes selective sampling possible in a real-time image acquisition setting.

6. REFERENCES

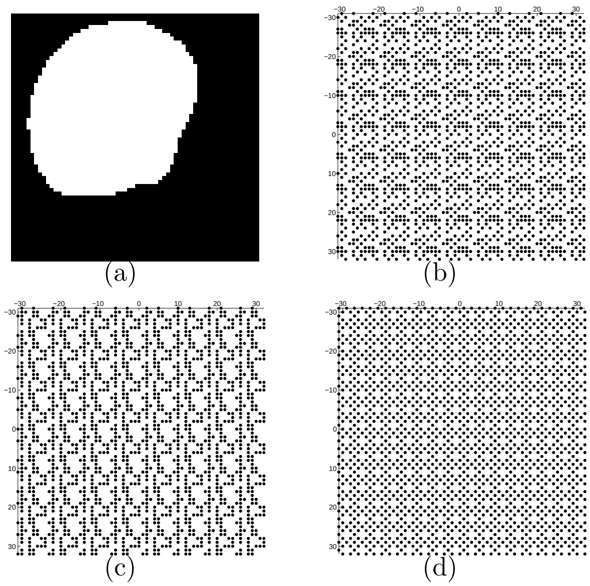

Figure 3: (a) ROS used in selection and reconstruction. (b) Optimized selection of shifted k-space arrays. (c) Randomly shifted arrays of k-space data. (d) Evenly spread shifted k-space arrays.

[1] Y. Gao and S. J. Reeves, "Optimal dithering of focal plane arrays in passive millimeter-wave imaging," *Optical Engineering*, submitted.

[2] S. K. Nagle and D. N. Levin, "A new class of sampling theorems for Fourier imaging of multiple regions," in *Proceedings of the 1998 IEEE International Conference on Image Processing*, vol. II, 1998.

[3] R. Venkataramani and Y. Bresler, "Further results on spectrum blind sampling of 2D signals," in *Proceedings of the 1998 IEEE International Conference on Image Processing*, vol. II, 1998.

[4] K. F. Cheung, *Advanced Topics in Shannon Samping and Interpolation Theory*, ch. A Multidimensional Extension of Papoulis' Generalized Sampling Expansion with the Application in Minimum Density Sampling, pp. 85–119. New York: Springer-Verlag, 1993.

[5] Y. Gao and S. J. Reeves, "Optimal sampling in array-based image formation," in *Proceedings of the 2000 IEEE International Conference on Image Processing*, September 2000.

[6] Y. Gao and S. J. Reeves, "Optimal k-space sampling in MRSI for images with a limited region of support," *IEEE Transactions on Medical Imaging*, to appear December 2000.