INSTANTANEOUS FREQUENCY ESTIMATION BASED ON THE ROBUST SPECTROGRAM

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ABSTRACT

Robust *M*-periodogram is defined for the analysis of signals with heavy-tailed distribution noise. In the form of a robust spectrogram (RSPEC) it can be used for the analysis of nonstationary signals. In this paper a RSPEC based instantaneous frequency (*IF*) estimator, with a time-varying window length, is presented. The optimal choice of the window length can resolve the bias-variance trade-off in the RSPEC based *IF* estimation. However, it depends on the unknown nonlinearity of the *IF*. The algorithm used in this paper is able to provide the accuracy close to the one that could be achieved if the *IF*, to be estimated, were known in advance. Simulations show good accuracy ability of the adaptive algorithm and good robustness property with respect to rare high magnitude noise values.

1. INTRODUCTION

A key-model of the *IF* concept is the complex-valued harmonic with a time-varying phase. It is an important model in the general theory of time-frequency (TF) distributions. This model has been utilized to study a wide range of signals, including speech, music, acoustic, biological, radar, sonar, and geophysical ones. An overview of the methods for the *IF* estimation, as well as the interpretation of the *IF* concept itself, is presented in [1]. One possible approach to the *IF* estimation is based on TF representations [2]-[4]. The SPEC is a commonly applied distribution within this approach.

In this paper we combine and develop two different ideas: the robust *M*-periodogram and the nonparametric approach [5]-[9] for selection of the time-varying adaptive window length in the corresponding periodogram. The robust *M*-periodogram is developed as a generalization of the standard periodogram for analysis of stationary signals corrupted with heavy tailed distribution noise [10]. Its form applied to the analysis of nonstationary signals will be referred to as the RSPEC. Recall that the heavy tailed noise is used as a model of an impulse noise environment [11]. The approach which exploits the intersection of confidence interval rule [12] was used in [13] for the standard periodogram based estimator with varying adaptive window length. It uses only the formula for the variance of the estimate, which does not require prior information about the *IF*. Simulations based on the discrete RSPEC show a good robustness and accuracy ability of the presented adaptive algorithm, as well as an improvement in the RSPEC based TF representation of signals with the nonlinear *IF*.

2. BACKGROUND THEORY

2.1. Robust spectrogram

Standard SPEC $I_S(t,\omega)$ definition, of a signal $x(t)$, is based on the standard short-time Fourier transform (STFT)

$$\hat{C}_h(t,\omega) = \frac{1}{\sum_n w_h(nT)} \sum_n w_h(nT) x(t+nT) e^{-j\omega nT} \quad (1)$$

$$I_S(t,\omega) = |\hat{C}_h(t,\omega)|^2$$

where the window $w_h(nT)=T\ w(nT/h)/h \geq 0$ has $h>0$ as a window length, and $\sum_n w_h(nT) \to 1$ as $h/T \to \infty$. Sampling interval is denoted by T.

The STFT $\hat{C}_h(t,\omega)$ may be derived as a solution of the following optimization problem [10]:

$$\hat{C}_h(t,\omega) = \arg\min_C J(\omega, C)\ , \quad (2)$$

where

$$J(t,\omega,C) = \sum_n w_h(nT) \cdot |x(t+nT) - C_h(t,\omega) e^{j\omega nT}|^2\ . \quad (3)$$

Here, the weighted square absolute error

$$F(e) = |e(nT)|^2 = |x(t+nT) - C_h(t,\omega) e^{j\omega nT}|^2 \quad (4)$$

is used as a loss function and minimized, by determining C. From $\partial J(t,\omega,C)/\partial C^* = 0$ definition (1) follows.

In [10] it has been shown that the loss functions of other forms than $F(e)=|e|^2$ can be more efficient in the optimization procedure (2). In particular, it has been shown that the loss function of the form $F(e)=|\mathrm{Re}\{e\}|+|\mathrm{Im}\{e\}|$ can produce very good results in the case of a signal corrupted with heavy tailed noise. The periodogram obtained using this loss function is called the robust *M*-periodogram. Its corresponding RSPEC is given in the form:

$$I_A(t,\omega) = |\hat{C}_h(t,\omega)|^2\ , \quad (5)$$

$$\hat{C}_h(t,\omega) = \arg\min_C J(t,\omega,C) \quad (6)$$

$$J(t,\omega,C) = \sum_n w_h(nT)\cdot[|\mathrm{Re}\{x(t+nT) - C_h(t,\omega)e^{j\omega nT}\}| + |\mathrm{Im}\{x(t+nT) - C_h(t,\omega)e^{j\omega nT}\}|].$$

By minimizing $J(t,\omega,C)$ we get a solution in the form

$$\hat{C}_h(t,\omega) = \frac{1}{\sum_n d(nT)}\sum_n d(nT)x(t+nT)e^{-j\omega nT},$$

$$d(nT) = \gamma(nT)/\sum_n \gamma(nT)$$

$$\gamma(nT) = \frac{w_h(nT)}{|x(t+nT) - C_h(t,\omega)e^{j\omega nT}|^2}\times$$

$$[|\mathrm{Re}\{x(t+nT) - C_h(t,\omega)e^{j\omega nT}\}| + |\mathrm{Im}\{x(t+nT) - C_h(t,\omega)e^{j\omega nT}\}|]$$

This is a set of nonlinear equations with unknown $C_h(t,\omega)$. It can be solved by using the following iterative procedure[10]:

Step 0. Initialization (standard STFT calculation):

$$C_h^{(0)}(t,\omega) = \frac{1}{\sum_n w_h(nT)}\sum_n w_h(nT)x(t+nT)e^{-j\omega nT} \quad (7)$$

$$\gamma^{(0)}(nT) = \frac{w_h(nT)}{|x(t+nT) - C_h^{(0)}(t,\omega)e^{j\omega nT}|^2}\times$$

$$\times[|\mathrm{Re}\{x(t+nT) - C_h^{(0)}(t,\omega)e^{j\omega nT}\}| + $$

$$+|\mathrm{Im}\{x(t+nT) - C_h^{(0)}(t,\omega)e^{j\omega nT}\}|]$$

(i) Step k, $k=1,2,..,K$:

$$C_h^{(k)}(t,\omega) = \frac{1}{\sum_n \gamma^{(k-1)}(nT)}\sum_n \gamma^{(k-1)}(nT)x(t+nT)e^{-j\omega nT},$$

$$\gamma^{(k)}(nT) = \frac{w_h(nT)}{|x(t+nT) - C_h^{(k)}(t,\omega)e^{j\omega nT}|^2}\times$$

$$\times[|\mathrm{Re}\{x(t+nT) - C_h^{(k)}(t,\omega)e^{j\omega nT}\}| + $$

$$+|\mathrm{Im}\{x(t+nT) - C_h^{(k)}(t,\omega)e^{j\omega nT}\}|]$$

with the stopping rule

$$\hat{k} = \min_k\left\{k: \frac{|C_h^{(k)}(t,\omega) - C_h^{(k-1)}(t,\omega)|}{|C_h^{(k-1)}(t,\omega)|} \le \eta,\ k \le K\right\},$$

where $\eta > 0$ and K are given.

(ii). Setting the RSPEC $I_A(t,\omega)$ as $I_A(t,\omega) = |C_h(t,\omega)|^2$, where $C_h(t,\omega) = C_h^{(\hat{k})}(t,\omega)$.

Experiments have shown fast convergence of the algorithm. Provided η=0.1 a usual number of iteration is between 3 and 5 and never exceeded 15.

2.2. IF Estimation

Consider now the problem of *IF* estimation, using the RSPEC, from the discrete-time observations

$$x(nT)=m(nT)+\varepsilon(nT), \text{ with } m(t) = Ae^{j\varphi(t)} \quad (8)$$

where n is an integer, T is a sampling interval and $\varepsilon(nT)$ is a complex-valued white noise $E(\varepsilon(nT))=0$, $E(|\varepsilon(nT)|^2)=\sigma^2$. By definition, the *IF* is the first derivative of the phase $\Omega(t) = \varphi'(t)$. Its estimate can be found as

$$\hat{\omega}_h(t) = \arg\max_{\omega\in Q_\omega} I_A(t,\omega) \quad (9)$$

where for a window $w_h(nT)$ there are N samples within the interval $Q_\omega \in [-\pi,\pi)$. Let us recall that the window $w_h(nT)$ implements the idea of nonparametric estimation of the time-varying $\Omega(t)$, fitted by a constant ω, within the narrow window around the time-instant t.

The asymptotic accuracy analysis of the robust *IF* estimator (9) has been done in [10]. According to that analysis, with the corresponding constraints, asymptotic formulae for the variance and bias of the *IF* estimation error $\Delta\hat{\omega}_h(t) = \Omega(t) - \hat{\omega}_h(t)$, are given by

$$\mathrm{var}(\Delta\hat{\omega}_h(t)) = V(F,G)\cdot\frac{T}{A^2h^3}W_\omega + o(T/h^3), \quad (10)$$

$$E(\Delta\hat{\omega}_h(t)) = B_\omega h^2\Omega^{(2)}(t) + o(h^2), \quad (11)$$

where $o(x)$ denotes a small value, such that $o(x)/x \to 0$ as $x \to 0$. The following notation has been used

$$W_\omega = \frac{\int_{-\infty}^{\infty} w^2(u)u^2du}{\left(\int_{-\infty}^{\infty} w(u)u^2du\right)^2}, \qquad B_\omega = \frac{\int_{-\infty}^{\infty} w(u)u^4du}{3!\int_{-\infty}^{\infty} w(u)u^2du} \quad (12)$$

$$V(F,G) = \int (F^{(1)}(v))^2dG(v)\Big/\left(\int F^{(2)}(v)dG(v)\right)^2. \quad (13)$$

where $T \to 0$, $h \to 0$, $T/h^4 \to 0$, $\Omega^{(1)}(t) \ne 0$, $\Omega^{(2)}(t) \ne 0$, G is the noise $\varepsilon(nT)$ pdf, and $F^{(1)}$ and $F^{(2)}$ are the derivatives of F.

Comments:

1. Let the noise distribution be Gaussian, $\varepsilon(nT) \sim N(0,\sigma^2/2)$, and the loss function be quadratic $F(e)=e^2$, then $V(F,G)=\sigma^2/2$. Substituting $V(F,G)=\sigma^2/2$ into (10) gives the known formula for the variance of the periodogram *IF* estimates. In particular, this formula can be obtained as a special case from more general results produced in [6]. In a similar way we obtain $V(F,G)=\pi\sigma^2/2$ for $F(e)=|e|$.
2. Note that $V(F,G)$ appears only in the formula for the variance. Thus, a choice of the loss function F influences only the variance of estimation but not the bias. The formulae for the bias are the same for the robust and nonrobust estimates [13].
3. Let us consider the mean squared error (*MSE*) of the estimate. From (10) and (11) follows that for small h the dominant terms of the *MSE* can be given in the form

$$E((\Delta\hat{\omega}_h(t)^2)) = V(F,G)TW_\omega/A^2h^3 + (B_\omega h^2\Omega^{(2)}(t))^2. (14)$$

Decrease of the window length h results in decrease of the bias and in increase of the variance, and vice versa. The optimal window width is given as

$$h_{opt}(t) = \left(3V(F,G)\cdot TW_\omega/4A^2(B_\omega\Omega^{(2)}(t))^2\right)^{1/7}.$$

It gives an optimal bias-variance trade-off, usual for nonparametric estimations. Optimal length depends on the signal-to-noise ratio A/σ_ε, the sampling interval T, noise distribution G, selected loss function F, and the second *IF* derivative $\Omega^{(2)}(t)$. Thus the optimal, or even reasonable choice of length h, depends on the *IF* second derivative $\Omega^{(2)}(t)$, which is naturally unknown because the *IF* itself is to be estimated.

3. ALGORITHM OF DATA-DRIVEN WINDOW LENGTH CHOICE

3.1. Basic idea ([13], [14])

The basic idea follows from the *IF* estimation error analysis. Namely, at least for the asymptotic case, the estimation error can be represented as a sum of the deterministic component (bias) and random component, with the variance given by (10). The estimation error can be written as

$$|\Omega(t)-\hat{\omega}_h(t)| \le |bias(t,h)| + \kappa\sigma(h)\,, \qquad (15)$$

with $\sigma^2(h)=\mathrm{var}(\Delta\hat{\omega}_h(t))$. Inequality (15) holds with probability $P(\kappa)$, where κ is the corresponding quantile of the standard Gaussian distribution $N(0,1)$. The usual choice κ=2 gives $P(\kappa)$=0.95. It follows from (11) that $|bias(t,h)|\to 0$ as $h\to 0$. Now, let $h=h_s$ be so small that

$$|bias(t,h_s)| \le \kappa\sigma(h_s)\,, \qquad (16)$$

then

$$|\Omega(t)-\hat{\omega}_{h_s}(t)| \le 2\kappa\sigma(h_s)\,. \qquad (17)$$

It is obvious that, for small h_s, all of the segments

$$D_s=[\hat{\omega}_{h_s}(t)-2\kappa\sigma(h_s),\hat{\omega}_{h_s}(t)+2\kappa\sigma(h_s)]\,, \qquad (18)$$

have a point in common, namely $\Omega(t)$. Consider an increasing sequence of h_s, $h_1<h_2<....$ Let h_{s+} be the largest of those h_s for which the segments D_{s-1} and D_s have a point in common. Let us call this window length h_{s+} 'optimal' and determine the *IF* estimates with data-driven optimal window length as $\hat{\omega}_{h_{s+}}(t)$. The basic idea behind this choice is as follows: If the segments D_{s-1} and D_s do not have a point in common it means that at least one of the inequalities (17) does not hold, i.e. the bias is too large as compared with the standard deviation in (16). Thus, the statistical hypotheses to be tested for the bias is given in the form of the sequence of inequalities (17) and the largest length h_s for which these inequalities have a point in common is considered as a bias-variance compromise, when the bias and variance are of the same order. Details on this two-segments intersection approach may be found in [14].

3.2. Algorithm

Let us initially assume that the amplitude A and the standard deviation σ of the noise are known. Let H be an increasing sequence of the window length values

$$H=\{h_s|h_1<h_2<h_3<...<h_J\}\,. \qquad (19)$$

In general, any reasonable choice of H is acceptable. In particular, the lengths with dyadic numbers $N_s=2N_{s-1}$ of observations within the window length, until the largest h_J is reached, will be assumed. This scheme corresponds to the radix-2 FFT algorithms. Note that the relation between the window length and the number of observations within that length is $h_s=NsT$. However, we want to emphasize that the minimum window size h_1 should not be too small (say h/T>20) in order to preserve the robustness property of algorithm with respect to the heavy-tailed distribution noise. The following steps are generated for each t.

1. The RSPEC is calculated for all $h_s\in H$. Thus, we obtain a set of RSPECs for a fixed time instant t, $\{I_A(\omega,t;h_s)\}, h_s\in H$. The *IF* estimates are found as

$$\hat{\omega}_{h_s}(t)=\arg[\max_{\omega\in Q_\omega} I_A(\omega,t;h_s)]\,. \qquad (20)$$

2. The upper and lower bounds of the confidence intervals D_s in (19) are built as follows

$$U_s(t)=\hat{\omega}_{h_s}(t)+2\kappa\sigma(h_s)\,,\quad L_s(t)=\hat{\omega}_{h_s}(t)-2\kappa\sigma(h_s)\,. \quad (21)$$

The variance $\sigma^2(h_s)$ is estimated by $\hat{\sigma}^2(h_s)=\hat{\sigma}^2(h_J)$ h_J^3/h_s^3, where $\hat{\sigma}^2(h_J)$ is the variance estimation obtained by using the widest window h_J, according to

$$\hat{\sigma}^2(h_J)=\frac{1}{N_J}\sum_{i=1}^{N_J}|x(t+iT)|^2-\hat{A}^2,$$

while $\hat{A}$ is the estimated amplitude of signal. It can be obtained applying the methods described [14] to signal $x(t+nT)/e(nT)$, where $e(nT)$ is the error (4).

3. The 'optimal' window length h_{s+} is determined as the largest $s=s_+$ (s=1,2,...,J) when

$$|\hat{\omega}_{h_{s-1}}(t)-\hat{\omega}_{h_s}(t)| \le 2\kappa(\sigma(h_{s-1})+\sigma(h_s))$$

is still satisfied, $\hat{h}(t)=h_{s+}(t)$ and $\hat{\omega}_{\hat{h}(t)}(t)$ is the adaptive *IF* estimator with the data driven window for a given t.

4. The RSPEC with the optimal window length is $I^+(\omega,t)=I_A(\omega,t;\hat{h}(t))$. Steps 1-4 are repeated for each considered instant t.

4. EXAMPLE

Consider now a signal with highly nonlinear *IF*

$$\Omega(t)=20\pi\,\mathrm{asinh}(12.5t)+128\pi \qquad (22)$$

The signal is embedded with a high amount of heavy tailed noise:

$$\varepsilon_H(nT)=1.5(\varepsilon_R^3(nT)+j\varepsilon_I^3(nT))/\sqrt{2}\,, \qquad (23)$$

where $\varepsilon_R(nT)$ and $\varepsilon_I(nT)$ are mutually independent white Gaussian noises $N(0,1)$. The non-noisy and noisy signals are shown in Figs.1a,b. In this case standard SPEC is useless for *IF* estimation, Figs.1c,d. Application of the RSPEC, Section II, along with the algorithm from Section III resulted in the adaptive window length that is shown in Fig.1e. MSE of the *IF* estimation, by using the RSPEC, versus window length is shown in Fig.1f. The straight line shows MSE for the *IF* estimation by using adaptive RSPEC. We may conclude that the adaptive estimation produces smaller MSE than *the best constant window length, which is also a priori unknown*. The RSPEC calculated using the adaptive window length is shown in Fig.1g. The adaptive *IF* is shown in Fig.1h. Obviously, for slow *IF* changes adaptive algorithm takes wider window length, while for faster changes it takes narrower window length, as expected.

5. CONCLUSION

The RSPEC as a time-varying form of the robust *M*-periodogram, with the varying adaptive window length, is developed. The intersection of confidence intervals rule is applied for varying window length selection. Simulation demonstrates that the new RSPEC gives the estimates of the varying *IF* which are strongly robust with respect to the noise having a heavy-tailed distribution. Note, that similar problems with another loss function is considered in [15].

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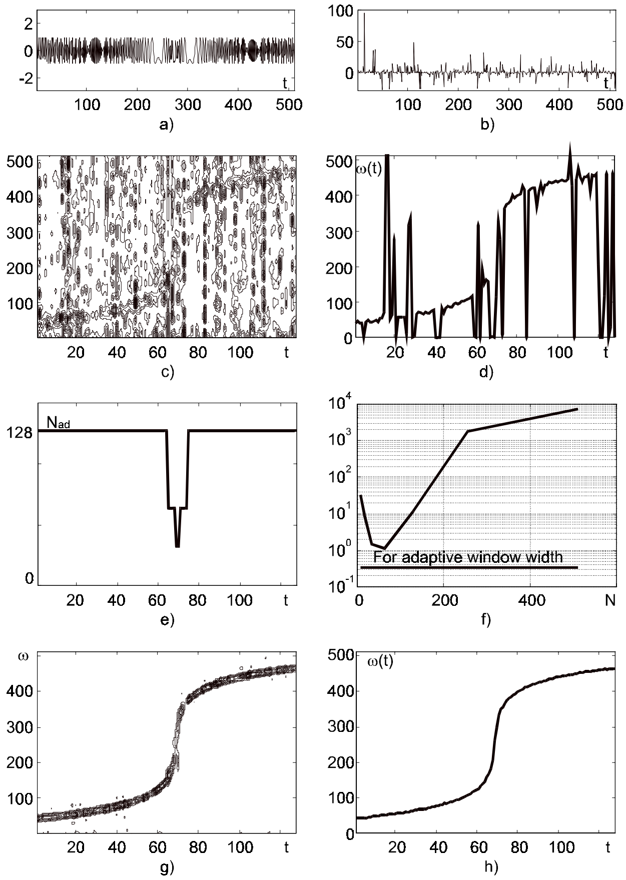

Fig.1. TF transforms and *IF* estimation of nonstationary signal with impulse noise: a) nonnoisy signal, b) noisy signal, c) standard SPEC with *N*=64, d) *IF* estimate based on the standard SPEC, e) adaptive window length in the RSPEC, f) MSE for various window lengths and adaptive window length of the RSPEC, g) RSPEC with adaptive window , h) *IF* estimate with the adaptive window RSPEC

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