MARKOVIAN MODEL OF THE ERROR PROBABILITY DENSITY AND APPLICATION TO THE ERROR PROPAGATION PROBABILITY COMPUTATION OF THE WEIGHTED DECISION FEEDBACK EQUALIZER

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ABSTRACT

A markovian model of the error probability density for decision feedback equalizer is proposed and its application to the error propagation probability computation is derived. The model is a generalization of the Lütkemeyer and Noll model proposed in [1]. It is obtained by the analysis of the gaussian mixture distribution of the errors which follows a Markov Process. The analysis of this process shows that the error propagation probability of the Weighted DFE [2] is less than the one of the classical DFE.

1. INTRODUCTION

Obviously, the number of services on heterogeneous wireless networks such as GSM, IS95, PDC, DECT and the future UMTS is increasing dramatically. In addition, one of the most challenging issues is interactive multimedia services over wireless networks. As a consequence, the bit-rate of these wireless networks is increasing dramatically. Consequently the spectrum efficiency of the modulation scheme should be more and more important. Being so, the sensitiveness of the transmitting signal to the multipath effects is also increasing.

To fight the multipath problem of wireless networks, some services have chosen multicarrier modulation as for instance DAB and DVB-T in Europe. In the case of monocarrier modulation we need powerful equalization techniques.

It is well known that MLSE equalizer is the best one, but it is also well known that its computational complexity depends on both the number of the constellation points and the length of the channel impulse response. Therefore MLSE is not usable in practice for high spectral efficiency modulation, and it is the main reason why people turn their attention to decision feedback equalizers (DFE). In fact it offers the best compromise between performances and complexity. DFE are well known for their superior performance compared to transversal equalizers, but due to their recursive structure (feedback loop), they can suffer from error propagation. This results in an overall mean square error (MSE) degradation.

This problem is well known and has already been addressed by many authors. In references [3, 4] the problem of bounds of this degradation has been addressed. More recently in reference [1] the authors established a probability state model for the calculation of the bit error rate degradation due to error propagation.

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This problem is very often overcome with the transmission of a known training data sequence. This training period is used both for the starting period (blind equalization) and for the tracking period. During this last period, the channel may change and the DFE in the decision directed (DD) mode may suffer from the error propagation. As a consequence, this training period should be transmitted regularly. This results in a loss of bit-rate

This error propagation problem is a major one and its thorough solution remains an open problem. An effective technique has recently been proposed in [5]. In that work the authors proposed a blind DFE by commuting, in a reversible way, both its structure and its adaptation algorithm, according to some measure of performance as, for instance, the mean square error (MSE). So, in this way, their DFE doesn't suffer from the error propagation problem.

In [2] and [6], we addressed the problem of error propagation as being the result of both the input of errors in the feedback filter and the divergence of the adaptive algorithm. This equalizer called weighted decision feedback equalizer (WDFE) offers the advantage to limit and solve the error propagation phenomenon.

In this paper for demonstrating the performance of the WDFE we define a markovian model of the error probability density for decision feedback equalizer and its application to the error propagation probability computation is derived. This model, a generalization of the Lütkemeyer and Noll model proposed in [1], takes into account only the filtering part of the equalizers. It is obtained by the analysis of the gaussian mixture distribution of the errors which follows a Markov process. This model is totally equivalent to the model recently published by Willink, *et. al* [7]. The analysis of this process shows that the error propagation probability of the Weighted DFE [6] is less than the one of the classical DFE. In fact we show that the transition matrix of the WDFE Markov model is better conditioned that the one of the classical DFE.

The first section presents the WDFE. The second section describes briefly the hypothesis, our model and the equivalence with the Willink, Wittke and Campbell model [7]. Finally, in the third section, we study the behavior of the transition matrix of the WDFE compared to the one of the classical DFE.

2. WDFE PRESENTATION

The weighted decision feedback equalizer is the classical DFE to which we add two devices (figure 1). The first device computes a reliability value (a likelihood information) for each output of the DFE. The second one uses this value in such a way as not to decide on errors in the feedback loop and also to minimize the effect of the errors in the LMS-DD algorithm. Two different ways for computing and using these reliability values have been presented in papers [2] and [6].

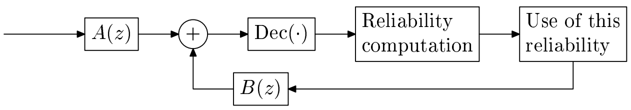

Fig. 1. WDFE scheme.

3. WDFE ERROR MODEL

3.1. WDFE filter equivalent structure

The filtering part of the WDFE could be synthesized by a DFE structure with a soft decision function U as describe in the figure 2.

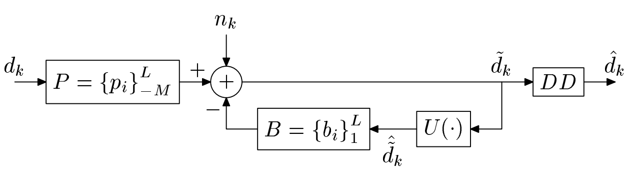

Fig. 2. WDFE equivalent structure.

This decision function is only used to compute the feed back symbols. The WDFE deals with three kinds of error because of the U function providing a new data estimation:

- $\tilde{e}_k = d_k - \hat{\tilde{d}}_k$ which is the new feedback error.
- $e_k = d_k - \tilde{d}_k$ which is the soft output error of the WDFE.
- $\hat{e}_k = d_k - \hat{d}_k$ which is the hard output error of the WDFE.

As the feedback errors $\tilde{e}_k$ and the decision error $\hat{e}_k$ are not equal, the results of the papers [1, 3, 8, 9, 4, 10, 11] have to be improved to match the WDFE particularities. Therefore, it is necessary to build a new model to achieve this goal.

3.2. The key equation

In the following, we assume that the channel is a additive white gaussian noise channel, and that the data are a white sequence of symbols uniformly distributed. Moreover, with the notations of figure 2, the feedback filter coefficients could be written as: $b_i = p_i + \delta_i$, and with $p_0 = 1 + \delta_0$, the output of the equalizer is then given by the equation

$$\tilde{d}_k = \sum_{i=-M}^{L} p_i d_{k-i} - \sum_{j=1}^{L} b_i \hat{\tilde{d}}_{k-i} + n_k. \qquad (1)$$

Let f_i be p_i for $i < 0$ and δ_i otherwise, the previous equation becomes

$$\tilde{d}_k - d_k = \sum_{i=-M}^{L} f_i d_{k-i} + \sum_{i=1}^{L} b_i \left(d_{k-i} - \hat{\tilde{d}}_{k-i} \right) + n_k. \qquad (2)$$

If the edge of the constellation is not taken into account, a function V could be defined as

$$d_{k-i} - U\left(\tilde{d}_{k-i}\right) = V\left(e_{k-i}\right). \qquad (3)$$

However, as seen later, this restriction could be removed. The existence of V simplifies the equation (2) in

$$e_k = -J(E_{k-1}) - I(D_k) - n_k, \qquad (4)$$

where $I(D_k)$ is the ISI part due to the B mismatch and the anti-causal part of the channel. E_k is the error state $[e_k \ldots e_{k-L+1}]$ and $J(E_k) = \sum_{i=1}^{L} b_i V\left(e_{k-i}\right)$ the ISI coming from the errors in the feedback filter.

3.3. The underlying Markov process

If we note $X|Y$ the random variable (r.v.) X conditioned on Y and f_X the probability density function (pdf) of the r.v. X, the key equation (4) gives

$$e_k | E_{k-1}, D_k \sim \mathcal{N}\left(-J(E_{k-1}) - I(D_k), \sigma_N^2\right). \qquad (5)$$

We could easily deal with the ISI coming from the data D_k by assuming

$$\Pr\left[e_k | E_{k-1}\right] = \sum_{D_k} \Pr\left[e_k | E_{k-1}, D_k\right] \Pr\left[D_k\right], \qquad (6)$$

and, as the data is a white uniformly distributed sequence, $\Pr\left[D_k\right] = 1/A^W$ with A the number of symbols of the modulation, and W the length of the channel P.

The relations (4), (5) define a L-order Markov sequence of the output error of the WDFE. The pdf of the error could be obtain by

$$f_{e_k}(x) = \sum_{D_k} \int_{\mathbb{R}^L} \frac{f_{e_k|E_{k-1},D_k}(x|Y, D_k)}{A^W} f_{E_{k-1}}(Y)\, dY. \qquad (7)$$

If we define g as the gaussian function with the variance σ_N^2, the previous equation becomes

$$f_{e_k}(x) = \sum_{D_k} \int_{\mathbb{R}^L} \frac{g\left(x + I(D_k) + J(E_{k-1})\right)}{A^W} f_{E_{k-1}}(Y)\, dY. \qquad (8)$$

These relations could be greatly simplified thanks to an approximation of the V function by a staircase function with N stairs defined by the parameters v_i and a_i:

$$\text{for } y \in [a_i; a_{i+1}] \;\; V(y) \approx v_i. \qquad (9)$$

The bounds a_i can be equal to $\pm\infty$ and the number of stairs is finite. Thanks to this approximation, the equation (8) becomes

$$f_{e_k}(x) = \frac{1}{A^W} \sum_{D_k} \sum_{i_1,\ldots,i_j=1}^{L} g\left(x + \sum_{j=1}^{L} b_i v_{i_j} + I(D_k)\right)$$
$$\times \int_{a_{i_1}}^{a_{i_1+1}} \cdots \int_{a_{i_L}}^{a_{i_L+1}} f_{e_{k-1} \ldots e_{k-L}}(y_1, \ldots, y_L)\, dy_1 \ldots dy_L. \quad (10)$$

We could define F_l the state of the WDFE which are the vector $[v_{i_1}, \ldots, v_{i_L}]$ and $\alpha_i^{(k)}$ the value of the multiple integration, then we have

$$f_{e_k}(x) = \sum_{i=1}^{N^L} \alpha_i^{(k)} \frac{1}{A^W} \sum_{D_k} g\left(x + Z\left(F_i\right) + I(D_k)\right), \quad (11)$$

where $Z(F_i) = \sum_{i_1,\ldots,i_L} \sum_j b_j v_{i_j}$ is the ISI coming from the error in the feedback filter using the approximation of the V function.

The last relation means that the pdf of the error is a gaussian mixture. Only the proportion of that mixture depends on the time k. If we re-inject this result into the equation (10), a time-relation between the proportion appears:

$$\alpha^{(k)} = Q \cdot \alpha^{(k-1)}, \quad (12)$$

with Q a $N^L \times N^L$ matrix depending on the channel, the feedback filter and the noise but not on the time.

This relation (12) highlights the Markov chain character of the WDFE error process. It is also possible to express the error probability at time k according to the proportion as a scalar product:

$$\begin{aligned} P_e^{(k)} &= 1 - \int_{-1}^{1} f_{e_k}(x)\, dx, \\ &= T^T \cdot \alpha^{(k)}. \end{aligned} \quad (13)$$

The last two equations (12) and (13) define the error model of the WDFE.

3.4. The modified error model

3.4.1. T weighting

For a M-PAM, the calculation of the error probability thanks to the equation (13) is not accurate because of the two symbols on the edge of the constellation. The integration is right for the $M-2$ inner symbols. For the 2 others ones, the integration domain is not $[-1;1]$ but $]-\infty;1]$ and $[-1;\infty[$ then the new T vector is obtained by

$$T_{\text{weighted}} = \frac{M-1}{M} T. \quad (14)$$

For modulation with a high spectral efficiency, the weighted factor tends toward 1 as expected.

3.4.2. Q weighting

For a 2-PAM DFE, the event $\tilde{e}_k = 2$ can appear only one time (for $d_k = 1$ and $\hat{d}_k = -1$). It is also the case for $\tilde{e}_k = -2$. But for $\tilde{e}_k = 0$ two cases are possible. This fact comes from the edge effect. In order to integrate it into the model, the lines of the transition matrix Q should be weighted according to it.

This modified model is for the DFE strictly equivalent to the Willink, Wittke, and Campbell model [7]. In fact, the proportion of the gaussian mixture is also the probability to be in a given state represented by the mean and the transition matrix are equal.

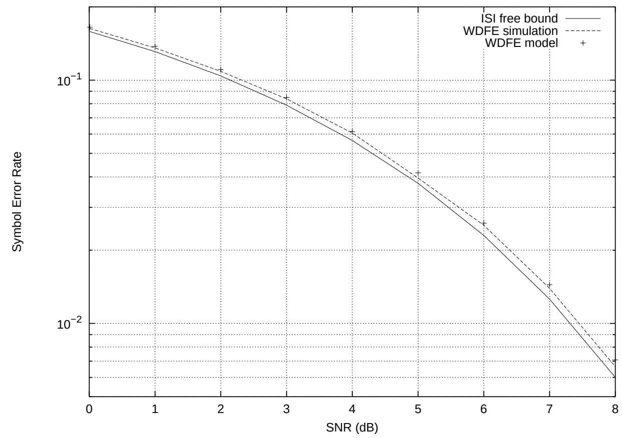

Fig. 3. Model validation for the WDFE rule 2.

To validate the model, a comparison between a Monte Carlo simulation and the model has been done (figure 3) with a channel set to $P = (1\ 0.2\ 0.2)$ and a backward filter set to $B = (0.24\ 0.24)$. The WDFE used the second rule [2, 6] approximated with 16 stairs for a 2-PAM.

4. WDFE/DFE COMPARISON

4.1. Introduction

As the transition matrices for the DFE and the WDFE do not have the same size, a comparison between their coefficients is not feasible. For this purpose, the WDFE Markov chain should be reduced into a DFE equivalent chain.

The states of the WDFE Markov chain holds the errors $\tilde{e}_k$ which is different from the DFE-like errors. But, if the $\tilde{e}_k$ is known, then the decision error is also known. The Markov chain of the WDFE could then be reduced in order to become a DFE-like chain. The reduction consists of grouping several WDFE states into one DFE states. This problem is the same as finding the transition probability $\Pr\left[\hat{e}_k | \hat{e}_{k-1} \ldots \hat{e}_{k-L}\right]$ knowing $\Pr\left[\hat{\tilde{e}}_k | \hat{\tilde{e}}_{k-1} \ldots \hat{\tilde{e}}_{k-L}\right]$.

4.2. Markov chain reduction

We define a Markov chain $\mathcal{M}$ with the transition matrix P and M states E_i. We note $k \in E_i$ the event: "we are in the state E_i at time k" and $E^{(k)}$ the probability of these events. We define also the K classes F_j which regroup some states E_i. E_i could be in only one group F_j. The grouping matrix $R_{K \times M}$ is then define as

$$R_{i,j} = \begin{cases} 1 & \text{if } E_j \text{ is in the class } F_i; \\ 0 & \text{otherwise.} \end{cases} \quad (15)$$

We search the transition matrix $Q^{(k)}$ of the reduced Markov chain $\mathcal{M}'$ defined by

$$Q_{i,j}^{(k)} = \Pr\left[k \in F_i | k-1 \in F_j\right]. \quad (16)$$

By developing the previous equation we found that it exists a matrix ${C^{(k)}}_{M\times K}$ such that

$$Q^{(k)} = RPC^{(k-1)}, \quad (17)$$

where

$$C^{(k)}_{i,j} = \Pr\left[k \in E_i | k \in F_j\right] = R_{j,i} \cdot \frac{\Pr\left[k \in E_i\right]}{\sum_{E_l \in F_j} \Pr\left[k \in E_l\right]}. \quad (18)$$

It is then trivial that $RC^{(k)} = I_{K\times K}$. The relations useful to switch between $\mathcal{M}$ and $\mathcal{M}'$ are

$$E^{(k)} = C^{(k)}F^{(k)} \qquad \text{and} F^{(k)} \qquad = RE^{(k)}. \quad (19)$$

4.3. WDFE reduction into a DFE

The previous discussion about the reduction of the WDFE Markov chain raises some difficulties, because the reduction transforms a time-independent chain into a time-dependent one. However it allows some numerical computation of criteria to compare the equalizers, like the eigenvalues which give an indication about the convergence speed of the process.

4.4. Two states reduction and propagation error probability

The reduction of the Markov chain gives us also an efficient criterion to compare the two kinds of DFE. In fact we could regroup the states into only two classes: the state $\mathcal{O}$ without decision error in the feedback filter and the state $\mathcal{E}$ which contains one or more errors. The error propagation probability P_{pe} is then the probability of the transition $\mathcal{E} \rightarrow \mathcal{E}$

$$P_{pe} = \Pr\Big[(\hat{e}_k \ldots \hat{e}_{k-L}) \neq 0 \;\Big|\; (\hat{e}_{k-1} \ldots \hat{e}_{k-L-1}) \neq 0\Big]. \quad (20)$$

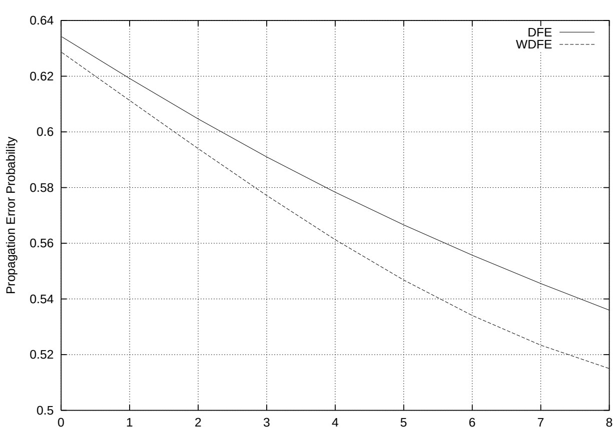

Fig. 4. Error propagation probability for the DFE and the WDFE.

The figure 4 presents the computation of the error propagation probability in the steady state with the same conditions as the previous simulation for different Signal to Noise Ratios.

5. CONCLUSION

The error probability model for the WDFE proposed in this paper is efficient enough to access to the error propagation probability of the DFE and confirms that this probability is less for WDFE than for classical DFE.

Future studies will investigate the behavior of this model with the adaptive algorithm of the WDFE.

6. REFERENCES

[1] C. Lütkemeyer and G. Noll, "A Probability State Model for the Calculation of the BER Degradation due to Error Propagation in Decision Feedback Equalizers," in *ITC*, Porto Carras, Greece, June 1998.

[2] J. Palicot, "A Weighted Decision Feedback Equalizer with Limited Error propagation," in *ICC 2000*, New Orleans, US, June 2000.

[3] D. Duttweiler, J. Mazo, and D. Messerschmitt, "An Upper Bound on the Error Probability in Decision-Feedback Equalization," *IEEE Trans. on Inform. Theory*, vol. IT-20, no. 4, pp. 490–497, July 1974.

[4] S. Altekar and N. Beaulieu, "Upper Bounds to the Error Probability of Decision Feedback Equalization," *IEEE Trans. on Inform. Theory*, vol. 39, no. 1, pp. 145–156, Jan. 1993.

[5] J. Labat and O. Macchi, "Adaptive Decision Feedback Equalization: Can You Skip the Training Period?," *IEEE Trans. on Com.*, vol. 46, no. 7, pp. 921–930, July 1998.

[6] J. Palicot and C. Roland, "A Weighted Decision Feedback Equalizer with complex decomposition of the reliability function," in *ICT 2000*, 2000.

[7] T. J. Willink, P. H. Wittke, and L. L. Campbell, "Evaluation of the Effect of Intersymbol Interference in Decision-Feedback Equalizer," *IEEE Trans. on Com.*, vol. 48, no. 4, pp. 629–635, Apr. 2000.

[8] W. Choy and N. Beaulieu, "Improved Bounds for the Error Recovery Times of Decision Feedback Equalization," *IEEE Trans. on Inform. Theory*, vol. 43, no. 3, pp. 890–902, May 1997.

[9] N. Beaulieu, "Bounds on Recovery Times of Decision Feedback Equalizers," *IEEE Trans. on Com.*, vol. 42, no. 10, pp. 2786–2794, Oct. 1994.

[10] J. Smee and N. Beaulieu, "Error-Rate Evaluation of Linear Equalization and Decision Feddback Equalization with Error Propagation," *IEEE Trans. on Com.*, vol. 46, no. 5, pp. 656–665, May 1998.

[11] N. Beaulieu, "The Evaluation of Error Probabilities for Intersymbol and Cochannel Interference," *IEEE Trans. on Com.*, vol. 39, no. 12, pp. 1740–1749, Dec. 1991.