SDP DESIGN PROCEDURE FOR ENERGY COMPACTION IIR FILTERS[1]

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ABSTRACT

In this paper we present a design method for optimal energy compaction IIR filters, where the numerator and denominator may have different degrees. The design is performed via iterative relaxations, where the numerator is optimized given the denominator, followed by optimization of denominator given the numerator. The two optimization problems involved are solved using semidefinite programming (SDP) techniques, where the real positiveness of the causal part of the product filter is formulated in two alternative ways: first using Kalman-Yakubovich-Popov (KYP) lemma, and second, by a less known parameterization[2][3], which we show to be more convenient numerically. Numerical results show the effectiveness of the proposed method and the improvements when compared with optimal FIR compaction filters or constrained IIR compaction filters (restricted to have allpass polyphase components).

1. INTRODUCTION

It was long observed that there is an IIR filter class which automatically fulfills the Nyquist(2) constraints, where the polyphase components are both allpass filters. Therefore in [4][6] the optimization of the optimum compaction gain of the filter was formulated in terms of poles (or polynomial coefficients) of the allpass filters involved. The structure based on allpass greatly simplifies the optimization problem, the set of Nyquist(2) constraints being automatically fulfilled. However if the structure of the IIR filter is arbitrary (i.e. the degrees of numerator and denominator are different) the Nyquist(2) constraints have to be enforced during optimization, making much more difficult the numerical solution for high degree filters.

Our work addresses the problem of arbitrary degree IIR filters design, for optimum energy compaction, under the constraint Nyquist(2). One of the most successful technique for the design of FIR optimal compaction filter is based on formulating the optimization of the product filter coefficients as a SDP problem[1][7], and then using the excitingly powerful tools developed in numerical analysis for solving SDP problems. We are using the same tools here for IIR energy compaction filter design.

[1] This work has been supported by Academy of Finland, project No. 44876 (Finnish Centre of Excellence Program (2000-2005))

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2. OPTIMUM ENERGY COMPACTION IIR FILTERS

Consider the IIR filter with the transfer function $H(z) = H_0(z^2)+z^{-1}H_1(z^2)$ where we can identify polyphase components in rational form

$$H(z) = \frac{B(z)}{A(z^2)} = \frac{B_0(z^2)}{A(z^2)} + z^{-1}\frac{B_1(z^2)}{A(z^2)}, \tag{1}$$

with

$$\begin{aligned} A(z) &= 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N} \\ B(z) &= b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_K z^{-K}. \end{aligned} \tag{2}$$

In this paper the polynomials will be denoted by upper case letters, while the vector of their coefficients will be denoted by the corresponding lower case letter. The product filter $G(z) = H(z)H(z^{-1})$ results in the form

$$G(z) = \frac{B(z)B(z^{-1})}{A(z^2)A(z^{-2})} = G_+(z) + G_+(z^{-1}),$$

where $G_+(z)$ is the causal and $G_+(z^{-1})$ the anticausal part of the product filter. The Nyquist(2) condition ($g_0 = 1$, $g_{2k} = 0$, $k \geq 1$) implies that $G_+(z^{-1})$ must have the form

$$G_+(z) = \frac{1}{2} + z^{-1}\frac{D\left(z^2\right)}{A\left(z^2\right)}, \tag{3}$$

where $D(z) = d_0 + d_1 z^{-1} + d_2 z^{-2} + \ldots + d_{N-1} z^{-(N-1)}$, and the connection between the polynomials $A(z)$, $B(z)$ and $D(z)$ is given by

$$\begin{aligned} &G^B(z) = B(z)B(z^{-1}) = A(z^2)A(z^{-2}) \\ &\quad + z^{-1}D(z^2)A(z^{-2}) + zD(z^{-2})A(z^2). \end{aligned} \tag{4}$$

For simplicity of notations we consider the degree of $D(z)$ to be $N-1$, but some trailing coefficients may be constrained to 0 in a or d, and consequently the true connection between the degrees of $D(z)$, $A(z)$ and $B(z)$ will result from (4).

The input spectrum is supposed to be rational and thus its autocorrelation sequence can be assumed of the form [4]

$$r_k = \sum_{i=1}^{n} S_i \rho_i^{|k|}. \tag{5}$$

The variance at the output of the filter $H(z)$ is evaluated as

$$\sigma^2 = r_0 + 2\sum_{k=0}^{\infty} g_{2k+1} r_{2k+1}$$

and using the expression (5) we obtain

$$\begin{aligned} \sigma^2 &= r_0 + 2\sum_{k=0}^{\infty} g_{2k+1} \sum_{i=1}^{n} S_i \rho_i^{2k+1} \\ &= r_0 + 2\sum_{i=1}^{n} S_i \left(G_+(\rho_i^{-1}) - \frac{1}{2} \right) \\ &= r_0 + 2\sum_{i=1}^{n} S_i \rho_i \frac{D\left(\rho_i^{-2}\right)}{A\left(\rho_i^{-2}\right)}. \end{aligned} \tag{6}$$

The optimum energy compaction problem can be formulated as

$$\begin{aligned} \max_{d,a} \quad & r_0 + 2\sum_{i=1}^{n} S_i \rho_i \frac{D\left(\rho_i^{-2}\right)}{A\left(\rho_i^{-2}\right)} \\ \text{s.t.} \quad & \frac{1}{2} + z^{-1}\frac{D\left(z^2\right)}{A\left(z^2\right)} \text{ is positive real} \\ & A(z) \text{ is stable} \end{aligned}$$

The criterion and constraints are strongly non-linear in parameters, and therefore general optimization techniques will suffer from the well known difficulties (e.g. local minima, slow convergence). Our strategy is to solve the problem by iterating two optimization stages, consisting each in finding one polynomial (d or a) given the other.

The IIR filter (1) can be obtained from the minimizing set (d, a) by performing the spectral factorization of (4).

3. OPTIMIZATION OF NUMERATOR FOR GIVEN DENOMINATOR

We deal first with the problem of optimizing the numerator $D(z)$ of (3), for fixed and stable denominator $A(z)$. The variance (6) depends linearly on the coefficients of $D(z)$, as it can be easily seen by inserting (5) in (6) to obtain

$$\sigma^2 = r_0 + 2\sum_{k=0}^{N-1} d_k \sum_{i=1}^{n} S_i \frac{\rho_i^{2k+1}}{A(\rho_i^{-2})} = r_0 + \beta^T d, \tag{7}$$

where at this stage β is a constant vector.

3.1. KYP parameterization

The constraint that $G_+(z)$ is positive real may be expressed by using the KYP lemma. We associate to $G_+(z)$ the controllable state-space realization (A, e_0, c, δ), with

$$A = \left[\begin{array}{cc} \multicolumn{2}{c}{-\tilde{a}^T} \\ I_{2N} & 0 \end{array} \right], \tag{8}$$

$$\tilde{a}^T = [0, a_1, 0, a_2, 0, \ldots, 0, a_N], \tag{9}$$

$$c^T = \tilde{d} = [d_0, 0, d_1, 0, \ldots, d_{N-1}, 0], \tag{10}$$

where e_0 is the first unit vector, $\delta = 1/2$ and I_k is the $k \times k$ unit matrix, and A is a stable matrix since $A(z)$ is supposed stable. The KYP lemma states that $G_+(z)$ is positive real if and only if a symmetric matrix $P \geq 0$ exists such that

$$Z = \left[\begin{array}{cc} 1 - e_0^T P e_0 & c^T - e_0^T PA \\ c - A^T P e_0 & P - A^T PA \end{array} \right] \geq 0. \tag{11}$$

We notice that all the variables, i.e. the coefficients of $D(z)$, are concentrated in the vector c. Consequently, (11) is a linear matrix inequality (LMI), as A, e_0 and δ are constants. In order to express the coefficients of the LMI, we introduce the following elementary matrices. Let E_{ik} be the symmetric matrix with ones in positions (i, k) and (k, i) and zeros elsewhere (indices start from zero), and denote $\Delta_{ik} = E_{i+1,k+1} - E_{ik}$. Then, the LMI (11) becomes

$$\begin{aligned} Z &= E_{00} + \sum_{k=0}^{N-1} d_k E_{2k+1,0} \\ &+ P_{00}\left(\Delta_{00} - \left[\begin{array}{cc} 0 & 0 \\ 0 & \tilde{a}\tilde{a}^T \end{array} \right] + \left[\begin{array}{cc} 0 & \tilde{a}^T \\ \tilde{a} & 0 \end{array} \right] \right) \\ &+ \sum_{i=1}^{2N-1} P_{i0} \left(\Delta_{i0} + \left[\begin{array}{c} 0 \\ \tilde{a} \end{array} \right] e_i^T + e_i[0, \tilde{a}^T] \right) \\ &+ \sum_{i=1}^{2N-1} \sum_{k=1}^{i} P_{ik} \Delta_{ik}. \end{aligned} \tag{12}$$

We may now formulate the optimization problem as

$$\begin{aligned} \max_{d,P} \quad & \beta^T d \\ \text{s.t.} \quad & Z \geq 0, \text{ with } Z \text{ as in } (12) \end{aligned} \tag{13}$$

This is a standard SDP problem in primal (or LMI) form and may be solved with dedicated routines. However, it has a large complexity, dictated by the number of variables in the LMI (as well as by the size of the LMI), since there are $O(N^2)$ extra variables introduced with P.

3.2. Alternative parameterization

As a faster alternative to (13) we can use a parameterization discussed in [2][3]. A symmetric polynomial $G^B(z)$ is positive on the unit circle if and only if a symmetric positive definite matrix Q exists such that

$$g_k^B = \mathrm{tr}\Theta_k Q, \tag{14}$$

where Θ_k is the elementary symmetric Toeplitz matrix with ones on the k'th diagonal and zeros elsewhere.

The coefficients of the causal part of the polynomial $G^B(z)$ from (4) are obviously linear combinations of the coefficients of $D(z)$, i.e.

$$g^B = Td + t, \tag{15}$$

where the matrix T and the vector t result easily from (4). The new formulation, equivalent to (13), is

$$\begin{aligned} \max_{d,g^B,Q} \quad & \beta^T d \\ \text{s.t.} \quad & g_k^B = \mathrm{tr}\Theta_k Q, \ k = 0 : 2N \\ & g^B = Td + t \\ & Q \geq 0, \end{aligned} \tag{16}$$

This is an SDP problem in dual (or equality) form. Its complexity is dictated by the number of equality constraints (and the size of Q, which exceeds only by one the size of the LMI from (13)); we have here only $O(N)$ equalities, which makes this approach faster than (13).

4. OPTIMIZATION OF THE DENOMINATOR FOR GIVEN NUMERATOR

We consider now the second problem, of optimizing $A(z)$ when $D(z)$ is given. Unfortunately, the matrix Z from (11) does not depend linearly on the coefficients of $A(z)$, and therefore in order to apply SDP we resort to an approximation, explained in the following.

We replace in (3) the term $1/A(z)$ by an (unknown) FIR approximation $W(z)$ of order $M > N$,

$$W(z) = w_0 + w_1 z^{-1} + w_2 z^{-2} + \ldots + w_M z^{-M}. \quad (17)$$

Certainly, by enlarging M a better approximation is obtained. Thus, instead of $G_+(z)$, we work with

$$G_+^W(z) = \frac{1}{2} + z^{-1} D(z^2) W(z^2). \quad (18)$$

The great benefit of this approach is that $G^W(z)$ is an FIR filter, for which an SDP problem can be formulated.

Due to (18), the optimization criterion becomes

$$\sigma^2 = r_0 + 2 \sum_{k=0}^{M} w_k \sum_{i=1}^{n} S_i \rho_i^{2k+1} D(\rho_i^{-2}) = r_0 + \beta_w^T w, \quad (19)$$

i.e. is linear in the variable w.

4.1. KYP parameterization

The degree of $G_+^W(z)$ is $N_w = 2(N+M) - 1$. The controllable state-space realization associated with $G_+^W(z)$ is $(A_w, e_0, \tilde{g}^W, \delta)$, with

$$A_w = \begin{bmatrix} 0 & 0 \\ I_{N_w} & 0 \end{bmatrix}, \quad (20)$$

$$\tilde{g}^W = [g_0^W, 0, g_1^W, 0, \ldots, 0, g_{N+M-1}^W]^T, \quad (21)$$

where $g^W = Vw$, with V the Toeplitz matrix

$$V = \begin{bmatrix} d_0 & 0 & \ldots & 0 \\ d_1 & d_0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & d_0 \\ d_{N-1} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & d_{N-1} \end{bmatrix}. \quad (22)$$

Denoting $v_k = Ve_k$ the columns of V, and $\tilde{v}_k$ the same columns with zeros inserted between each two elements, i.e.

$$\tilde{v}_k^T = [0, v_{0k}, 0, v_{1k}, 0, \ldots, 0, v_{N+M-1,k}], \quad (23)$$

the matrix Z from the KYP lemma (11) for the state space realization $(A_w, e_0, \tilde{g}^W, \delta)$ becomes

$$\begin{aligned} Z &= E_{00} + \sum_{k=0}^{M} w_k (\tilde{v}_k e_0^T + e_0 \tilde{v}_k^T) \\ &+ \sum_{i=0}^{N_w - 1} \sum_{k=0}^{i} P_{ik} \Delta_{ik}. \end{aligned} \quad (24)$$

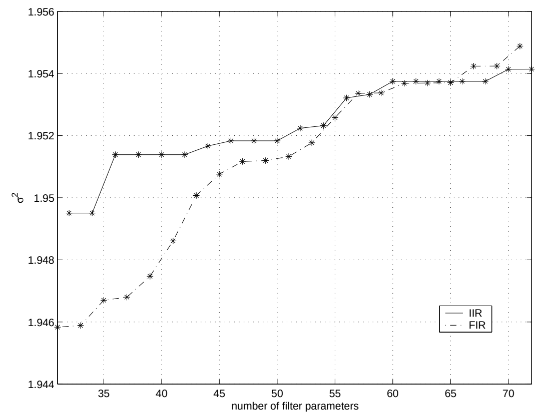

Fig. 1. The energy compaction criterion (σ^2) for optimal FIR[1] and optimal IIR filters designed with the new method designed for a AR(16) input signal.

Finally, we obtain the optimization problem

$$\begin{aligned} \max_{w,P} \quad & \beta_w^T d \\ \text{s.t.} \quad & Z \geq 0, \text{ with } Z \text{ as in } (24) \end{aligned} \quad (25)$$

4.2. Alternative parameterization

We apply now the same parameterization from Section 3.2, this time for $G_+^W(z)$, which is the causal part of a polynomial positive on the unit circle. We obtain

$$\begin{aligned} \max_{w, g^W, Q} \quad & \beta_w^T w \\ \text{s.t.} \quad & g_k^W = \text{tr}\Theta_{2k+1} Q, \ k = 0 : N+M-1 \\ & \text{tr}Q = 1, \ \text{tr}\Theta_{2k} Q = 0, \ k = 1 : N+M-1 \\ & g^W = Vw \\ & Q \geq 0 \end{aligned} \quad (26)$$

Again, this problem is equivalent to (25), but has a lower complexity.

5. THE ITERATIVE OPTIMIZATION PROCEDURE

Given the coefficients $\{S_i, \rho_i\}_{i=1:n}$, which generate the covariance in (5), we iterate for the parameters in (3).

0. Initialize: $k = 0$, $D^{[0]}(z^2) = 1$, $A^{[0]}(z^2) = 1$.

Iterate:

1. Determine the new $A^{[k+1]}(z^2)$ as follows:

1.1 Solve for w the optimization problem (26).

1.2 Obtain a stable polynomial $A(z)$ as approximate solution of

$$W(z^2) = \frac{1}{A(z^2)}$$

2. Determine the new $D^{[k+1]}(z^2)$ as follows:

2.1 Solve for d the optimization problem (16).

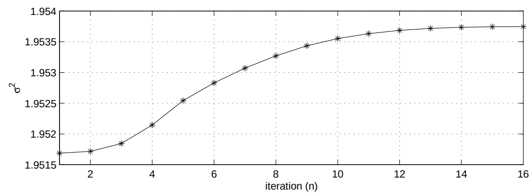

Fig. 2. The values of the criterion during the maximization with the new method designed for a AR(16) input signal.

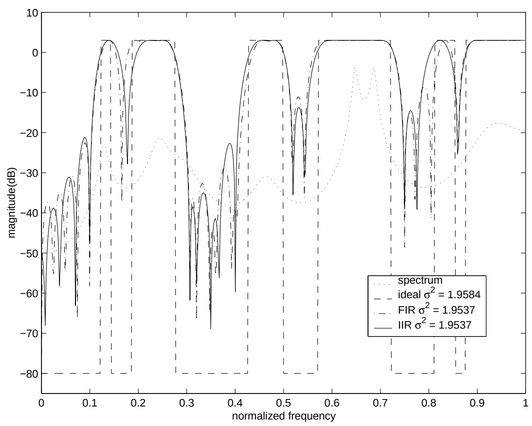

Fig. 3. Frequency responses: optimal FIR, optimal IIR (with the new method), ideal filter and input spectrum.

Until $\{\sigma^2\}^{[k+1]}$ (given by (16)) is not significantly larger than $\{\sigma^2\}^{[k]}$ (at previous iteration).

At the end of the iterations, the spectral factorization in (4) will provide the numerator $B(z)$ of the energy compaction IIR filter $H(z) = B(z)/A(z^2)$.

6. EXPERIMENTAL RESULTS

For simulations we considered as input an $AR(16)$ process. The spectrum of the input is displayed with dotted line in Figure 3. In the iterative procedure of Section 5, the SDP problems were solved by using SDPT3 package[5].

In Figure 2 we illustrate the behaviour of the iterative algorithm for the case of following structure of IIR: deg$D(z) = 23$ and deg$A(z) = 10$. It can be seen that after the first ten iterations the improvements become negligible. Generally, in all tested cases, most of the improvements have been obtained in the first two iterations.

In Figure 1 we present comparisons of σ^2 obtained with the optimal FIR[1] and optimal IIR filters designed with the new method. One can compare easily the small gain in performance for IIR filters having the same number of parameters as the FIR filter. We illustrate in Figures 3 and 4 the frequency response of optimal FIR and optimal IIR, at the same criterion value $\sigma^2 = 1.9537$. Even if the same value of σ^2 is obtained, different kinds of approximation of the ideal filter can be observed (e.g. steepness of transitions, sidelobes). We note that the allpass restricted IIR filter [4] having twelve parameter gives the compaction gain 1.931.

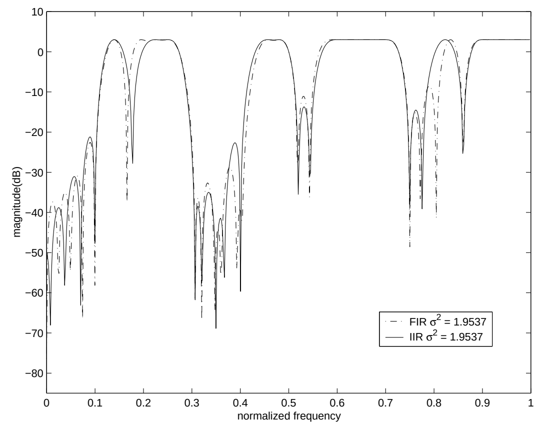

Fig. 4. Only two plots from Figure 3: frequency response of optimal FIR and optimal IIR.

We finally mention only qualitatively the ranking of the new methods in terms of running times. The new method based on KYP (13),(25), is about 5 times slower than the new method (16),(26) (based on the alternative parameterization) for the length of the filters deg$D(z) = 10$ and deg$A(z) = 10$; moreover for the case deg$D(z) = 23$ and deg$A(z) = 10$ the KYP variant could not be run on the PC we used for experiments (Pentium III at 700MHz) while the alternative parametrization runs in about 1 min/ iteration.

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