SPACE-TIME SIGNALING AND FRAME THEORY

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ABSTRACT

Wireless systems with multiple transmit and receive antennas (MIMO systems) provide high capacity due to the plurality of modes available in the channel. Previous code designs for MIMO systems have focused primarily on multiplexed signaling for high data rate or diversity signaling for high link reliability. In this paper, based on previous work reported in [1, 2], and using results from frame theory, we present a MIMO space-time code design which bridges the gap between multiplexing and diversity and *performs well both in terms of ergodic capacity as well as error probability*. In particular, we demonstrate that designs performing well from an ergodic capacity point of view do not necessarily perform well from an error probability point of view. Simulations illustrate performance of the proposed codes in narrowband MIMO Rayleigh fading channels.

1. INTRODUCTION

Multiple-input multiple-output (MIMO) antenna systems, i.e., wireless systems with multiple transmit and receive antennas, have high capacity both in theory [3, 4, 5] and in practical implementations [6]. Signaling, or coding, for MIMO systems, however, has been primarily limited to *multiplexing* [7, 3] and *diversity* [8, 9] modes of operation. Multiplexing refers to transmission of independent streams of data from each transmit antenna while diversity refers to transmission of multiple different "replicas" of the same data on each transmit antenna. Multiplexing transmission provides high throughput whereas diversity transmission aims at high link reliability. In practice, it is desirable to have the flexibility to distribute the available degrees of freedom between multiplexing and diversity. Therefore, code designs that fall between multiplexing and diversity modes of operation are needed.

Contributions. In this paper we use the linear coding framework introduced in [1], and later extended in [2], in which space-time codeword matrices are obtained as linear combinations of certain basis matrices with the expansion coefficients being scalar complex-valued data symbols. Therefore, different linear combinations of scalar data symbols are transmitted on each antenna. We derive the ergodic capacity of the effective channel induced by this signaling scheme and compute an upper bound on its pairwise error probability (PEP) for the independent identically distributed (i.i.d.) Rayleigh fading case. Based on a capacity bound, we derive optimum coding matrices. The resulting criterion is shown to be equivalent to requiring that a certain stacked codeword matrix yields a tight frame [10]. To ensure good performance in terms of symbol error rate as well we impose a side constraint taking into account the PEP.

Relation to previous work. Previous work has focused primarily on pure spatial multiplexing transmission [3, 7] or transmit diversity (i.e., space-time coding) [1, 8, 9]. Diversity versus multiplexing comparisons can be found in [11] in terms of instantaneous probability of error. Alternative code designs for the framework in [1] have also appeared in [2] based on ergodic capacity. Compared with [1, 2], we take into account *both error probability and ergodic capacity*, we present an upper bound on PEP, and we derive a novel criterion for code design which relates to tight frames [10]. Simulations show that while codes can perform equally well from an ergodic capacity point of view, there can be significant differences from an error rate point of view. Thus, taking into account error probability is necessary to guarantee reliability of the proposed codes.

Organization of the paper. The rest of this paper is organized as follows. In Section 2, we describe the signaling scheme, derive its ergodic capacity, and present an upper-bound on the PEP. In Section 3, we present a code design criterion based on an upper bound on ergodic capacity and show that such codes are related to tight frames. Section 4 provides Monte-Carlo simulations of some proposed codes. Finally, Section 5 contains our conclusions.

2. CODES, CAPACITY, AND PEP

In this section, we shall describe the framework for space-time code design for MIMO systems based on linear matrix modulation.

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Code Description. Let $\{s_n\}_{n=0}^{N-1}$ denote a block of N complex symbols (possibly coded) with R bits per symbol. A matrix modulation code for a system with M_t transmit antennas, M_r receive antennas, and block length of T time symbols is described as follows. Define the basis matrices of the code as the set of $M_t \times T$ matrices $\{\mathbf{M}_n\}_{n=0}^{N-1}$ which are possibly complex. The transmitted codeword will be the $M_t \times T$ matrix constructed by taking a linear combination of these matrices according to

$$\mathbf{M}(s_0,\, s_1,\, \ldots\, ,\, s_{N-1}) \quad := \quad \sum_{n=0}^{N-1} \mathbf{M}_n s_n. \qquad (1)$$

One column of $\mathbf{M}$ is transmitted per symbol period over T periods. Note that the effect of the code is to spread every symbol across every transmit antenna in every time slot. The approach in (1) differs from that in [1, 2] in that we do not take additional linear combinations as a function of the conjugate data symbols s_n^*.

While (1) provides intuition, an alternative representation leads to a more efficient analytical representation. Let the $M_t \times N$ matrices $\mathbf{X}_t$ be given by

$$\mathbf{X}_t \;:=\; [\mathbf{M}_{0,t} \;\; \mathbf{M}_{1,t} \;\; \ldots \;\; \mathbf{M}_{N-1,t}]$$

for $t = 0, 1, \ldots, T-1$, where $\mathbf{M}_{i,t}$ denotes the t-th column of the matrix $\mathbf{M}_i$. Define[1] $\mathbf{s} := [s_0,\, s_1,\, \ldots, s_{N-1}]^T$. In what follows we assume that the channel $\mathbf{H}$ is an $M_r \times M_t$ matrix of complex gaussian $\mathcal{CN}(0,1)$ coefficients, with coherence time T symbols. The output of the t-th transmission can be written as

$$\mathbf{y}_t \;\;=\;\; \sqrt{E_s}\,\mathbf{H}\,\mathbf{X}_t\,\mathbf{s} \;+\; \mathbf{v}_t,\; t = 0, 1, \ldots, T-1, \; (2)$$

where $\mathbf{y}_t$ is an $M_r \times 1$ received data vector and $\mathbf{v}_t$ is an i.i.d. complex gaussian spatially white noise vector distributed as $\mathcal{CN}(\mathbf{0}, N_o \mathbf{I}_{M_r})$.

For convenience stack T observations at the receiver. Let $\mathbf{y} := [\mathbf{y}_0^T \;\; \mathbf{y}_1^T \ldots \mathbf{y}_{T-1}^T]^T$, $\mathbf{v} := [\mathbf{v}_0^T \;\; \mathbf{v}_1^T \ldots \mathbf{v}_{T-1}^T]^T$, $\mathcal{X} := [\mathbf{X}_0^T \;\; \mathbf{X}_1^T \;\ldots\; \mathbf{X}_{T-1}^T]^T$, and $\mathcal{H} := \mathbf{I}_T \otimes \mathbf{H}$ where $\otimes$ is the Kronecker product. Then (2) becomes

$$\mathbf{y} \;\;=\;\; \sqrt{E_s}\mathcal{H}\,\mathcal{X}\,\mathbf{s} \;+\; \mathbf{v}. \qquad (3)$$

We furthermore assume the following normalizations[2]: (i) $\mathcal{E}\{|\mathbf{s}_n|^2\} = 1$ for all n and (ii) $\mathrm{tr}(\mathcal{X}^H \mathcal{X}) = T$ to make comparisons fair with other coded/uncoded systems. Effectively (3) is an input-output relationship of an $M_r T \times N$ MIMO system with equivalent channel $\mathcal{H}\,\mathcal{X}$.

Capacity of Proposed Signaling Scheme. Let $\mathbf{Q}$ be the $N \times N$ covariance matrix corresponding to the zero-mean input vector $\mathbf{s}$. The ergodic capacity in bits/s/Hz of the system in (3) for a given $\mathcal{X}$ can be written as

$$C \;=\; \max_{\mathrm{tr}(\mathbf{Q})=T} \frac{1}{T}\mathcal{E}_H \left\{ \log\det\left(\mathbf{I}_{M_r T} + \frac{E_s}{N_o}\,\mathcal{H}\mathcal{X}\mathbf{Q}\mathcal{X}^H\mathcal{H}^H \right) \right\},$$

where $\mathcal{E}_H$ stands for expectation with respect to the channel. The normalization by $1/T$ accounts for the spreading of information across time.

In the code design problem we seek $\mathcal{X}\mathbf{Q}\mathcal{X}^H$ which maximizes ergodic capacity. Since there is no channel knowledge at the transmitter we set $\mathbf{Q} = \mathbf{I}$. The capacity of the optimum code design is written as

$$C \;=\; \max_{\mathrm{tr}(\mathcal{X}^H\mathcal{X})=T} \frac{1}{T}\mathcal{E}_H \left\{ \log\det\left(\mathbf{I}_{M_r T} + \frac{E_s}{N_o}\,\mathcal{H}\mathcal{X}\mathcal{X}^H\mathcal{H}^H \right) \right\}. \qquad (4)$$

Observe from (4) that the number of modes available in the system is $\min(M_t T, M_r T, N)$. In what follows we assume that $N \leq M_t T$ thus $\mathcal{X}$ is square or tall.

A few comments on (4) in relation to other block modulation schemes proposed in the literature are in order. For $M_r = 1$, $M_t = 2$, and $T = 2$, $N = 2$ gives the maximum number of modes which is achieved by the real-symbol version of the Alamouti transmit diversity scheme [9]. For $M_r = 2$, however, there are $N = 4$ modes available thus the Alamouti scheme incurs a loss in capacity [12], whereas simple spatial multiplexing [7] achieves full capacity.

PEP of Proposed Signaling Scheme. Whilst the ergodic capacity captures the performance achieved as the number of independent fading intervals over which channel coding is performed goes to infinity, the error probability more accurately characterizes performance in practical systems with coding over a finite number of fading intervals. In particular, the error probability captures the so-called diversity advantage of a code which is not revealed by the ergodic capacity expression in (4).

Unfortunately, the exact probability of error is difficult to calculate. We instead examine the performance through the Chernoff upper bound on the PEP. This bound is a common tool for code design in diversity systems (e.g., see [8]) and can be used to upper bound the probability of symbol error.

Let $\mathbf{s}$ denote the transmitted symbol vector. For a given channel realization $\mathbf{H}$, the probability that the receiver decides erroneously in favor of the vector $\hat{\mathbf{s}}$, assuming the maximum likelihood (ML) receiver, is given by

$$P(\mathbf{s} \to \hat{\mathbf{s}} \mid \mathbf{H}) = Q\left(\sqrt{\frac{E_s}{2N_o} d_{\mathbf{e}}^2} \right), \qquad (5)$$

where $d_{\mathbf{e}}^2 = \|\mathcal{H}\mathcal{X}(\mathbf{s} - \hat{\mathbf{s}})\|^2$. The average over all channel realizations of the right-hand-side of (5) can now be upper bounded as

$$P(\mathbf{s} \,\to\, \hat{\mathbf{s}}) \;\leq\; \frac{1}{\left| \mathbf{I}_{M_t M_r} + \frac{E_s}{4N_o}\mathbf{I}_{Mr} \otimes \mathbf{R} \right|},$$

[1] The superscripts ^{T}, ^{H}, and * stand for transposition, conjugate transposition, and elementwise conjugation, respectively.

[2] $\mathcal{E}$ denotes the expectation operator.

where

$$\mathbf{R} \;:=\; \sum_{t=0}^{T-1} \mathbf{X}_t^* \mathbf{e}^* \mathbf{e}^T \mathbf{X}_t^T \quad (6)$$

with $\mathbf{e} = \mathbf{s} - \hat{\mathbf{s}}$ and $|\mathbf{A}|$ denotes the determinant of the matrix $\mathbf{A}$.

The *diversity order* of the code is defined as

$$M_r \min_{\mathbf{e} \in \mathbf{E}} \operatorname{rank} \left(\sum_{t=0}^{T-1} \mathbf{X}_t^* \, \mathbf{e}^* \, \mathbf{e}^T \, \mathbf{X}_t^T \right) \quad (7)$$

where $\mathbf{E}$ is the set of all possible error vectors. From (7), the diversity order is upper bounded by $M_r \min\,(M_t, T)$ motivating choice of $T \geq M_t$ to maximize diversity gain. The *coding advantage* is defined as the smallest product of the nonzero eigenvalues of $\mathbf{R}$ in (6). The coding advantage relates to the SNR improvement of the resulting code and should be as large as possible for a given diversity order. Larger T gives more degrees of freedom in the design but typically increases decoding complexity.

3. CODES AND FRAMES

In this section, we formulate criteria for code design, i.e., criteria for selecting the matrix $\mathcal{X}$. Denote an $(M_t,\ M_r,\ T,\ N,\ \mathcal{C})$ code as one designed for an $M_r \times M_t$ channel, using block length T, transmitting N symbols taken from constellation $\mathcal{C}$ (e.g., 4QAM). Define the *multiplexing order* of such a code as $\min(M_t T,\ M_r T,\ N)/T$. The multiplexing order describes the effective number of modes used per symbol time.

Proposed Code Design Criteria. A closed form expression for the optimum $\mathcal{X}$ in (4) for the general case seems difficult to obtain. Therefore, we optimize instead an upper bound on the capacity. Application of Jensen's inequality [13] to (4) yields

$$C \quad \leq \max_{\operatorname{tr}(\mathcal{X}^H \mathcal{X}) = T} \frac{1}{T} \log \det \left(\mathbf{I}_N + \frac{E_s}{N_o}\, M_r \, \mathcal{X}^H \mathcal{X} \right) \quad (8)$$

which is maximized by $\mathcal{X}^H \mathcal{X} = T/N\, \mathbf{I}_N$ for $N \leq M_t T$. Unfortunately, while (8) gives insight into capacity maximization, it does not guarantee good performance in terms of error probability. Examples are provided in the simulations section. Since the upper bound on the PEP is a good predictor of code performance at high SNR we incorporate this to propose the following design criterion.

> **Design Criterion**. *For $(M_t, M_r, T, N, \mathcal{C})$, maximize the diversity order (7), and then the coding advantage, subject to $\mathcal{X}^H \mathcal{X} = T/N\, \mathbf{I}_N$.*

Coefficients of these matrices can be found through nonlinear optimization or through random search techniques. Details are provided in [14].

Relation with Frame Theory. Design of appropriate codes is simplified by recognizing that $\mathcal{X}^H \mathcal{X} = T/N\, \mathbf{I}_N$ is achieved by any appropriately scaled tight frame for N-dimensional space with redundancy $M_t T/N$ [10]. Here, the rows of $\mathcal{X}$ constitute the frame elements. The connection with frame theory gives the interpretation that we transmit the coefficients of the frame expansion, given by $\mathcal{X}\mathbf{s}$ as *opposed to the symbols themselves*.

Any tight frame $\mathcal{X}$ achieves the upper bound on capacity (8). For example, if $\{\mathbf{X}_t\}_{t=0}^{T-1}$ are all tight frames (requires $N \leq M_t$) with frame bound $1/N$, then $\mathcal{X}$ is a tight frame with frame bound T/N. We emphasize, however, that even though all tight frames perform more or less equally well from an ergodic capacity point of view, there can be significant differences in their error rate performance.

It is interesting to observe that in some cases tight frames achieve the exact capacity.

Lemma 1 *For $N = M_t T$, tight frames achieve capacity. Proof: In this case, $\mathcal{X}\mathcal{X}^H = T/N \mathbf{I}_{M_t T}$, thus the capacity in (4) becomes $C = \mathcal{E}_H\{\log\det(\mathbf{I}_{M_r} + E_s/N_o\, \mathbf{H}\mathbf{H}^H)\}$, the capacity of the $M_r \times M_t$ channel [3, 15].*

Note that in this case the redundancy equals 1 and hence the tight frame is an orthogonal system. Unfortunately, diversity advantage is lost no matter how many transmit antennas are used, due to the lack of additional degrees of freedom.

In addition to interpretation, frame theory provides a rich area to design code matrices. For instance, given any random matrix $\hat{\mathbf{X}}$, a frame can be constructed from an appropriately scaled $\hat{\mathbf{X}}(\hat{\mathbf{X}}^H \hat{\mathbf{X}})^{-1/2}$. In frame-theoretic language this amounts to constructing a tight frame from a given nontight frame. Therefore, one procedure for designing codes which perform well both from an ergodic capacity point of view and from an error probability point of view, is to randomly select an $\hat{\mathbf{X}}$, compute the corresponding tight frame, evaluate the PEP rank criterion and proceed until a sufficiently good code is found. We simulate codes found using this method in the sequel. Other techniques relying on optimization are explored in [14].

4. SIMULATIONS

In this section, we provide performance analysis of some example codes. We consider two designs for a $(3,\, 2,\, 2,\, 4,\, 4QAM)$ code based on the random search technique. Let $X \times Y$ denote a system with X transmit antennas and Y receive antennas. We illustrate performance of a code with good PEP performance, one with bad PEP performance, uncoded 2×2 spatial multiplexing, and the Alamouti 2×2 block code [9] (with $16QAM$). The receiver implements ML decoding with perfect channel knowledge. In Fig. 1 we plot the ergodic capacity of a 3×2 system and compare with the equivalent system induced by the proposed codes. We see that all the proposed codes have similar performance

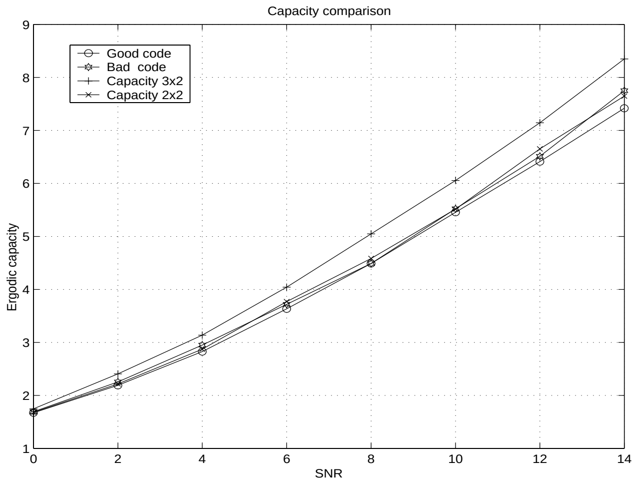

Fig. 1. Comparison of 3×2 ergodic capacity with that induced by two different $(3, 2, 2, 4, 4QAM)$ codes.

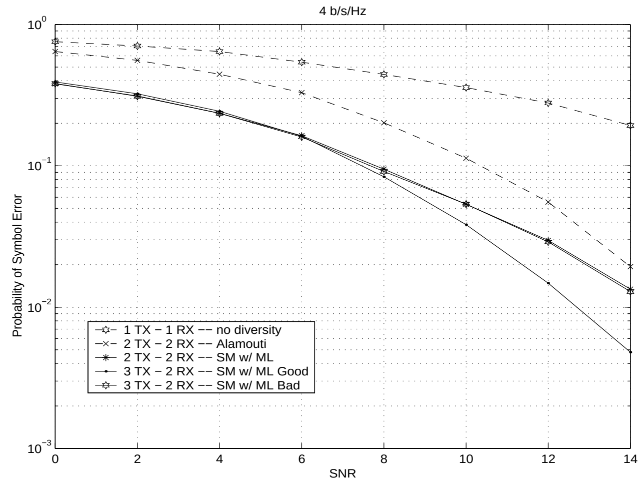

Fig. 2. Comparison of two potential $(3, 2, 2, 4, 4QAM)$ codes with other spatial signaling schemes.

in terms of ergodic capacity. The difference with uncoded 3×2 capacity and similarity to the 2×2 capacity is due to the fact that $N < TM_t$.

Alternatively, in Fig. 2 we estimate the symbol error probability of the proposed codes via 750 Monte Carlo simulations of bursts with 100 symbols per burst. First, note that use of $T = 2$ enables us to capture additional diversity advantage (of order four) from the third transmit antenna which is not available to the 2×2 spatial multiplexing system (which achieves diversity advantage of order two). Second, observe that without PEP optimization, codes that perform well from an ergodic capacity point of view may be found which do not give full diversity advantage. Thus PEP based optimization can dramatically improve code performance in practical systems. We explore this point further in [14].

5. CONCLUSIONS

For the linear matrix-modulation framework proposed in [1, 2], we derived a *code design criterion motivated by both the ergodic capacity and the probability of error*. We established a relation between the new class of codes and tight frames, and used results from frame theory to design codes. MIMO space-time codes designed according to our criterion perform well both from an ergodic capacity point of view and from an error probability point of view. This is important because, as was illustrated, two code designs which are equivalent in terms of ergodic capacity may have significantly different error rate performance.

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