COD: BLIND PATH SEPARATION WITH LIMITED ANTENNA SIZE

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ABSTRACT

This paper proposes a generalized *multipath separability condition* for subspace processing and derives a novel **COD** (Combined Oversampling and Displacement) algorithm to utilize both spatial and temporal diversities for path separation and DOA estimation. A unique advantage lies in its ability to cope with the situation where the number of multipaths is much larger than that of antenna elements, which arises in many practical situations. The traditional data matrix or any of its horizontally expanded versions cannot yield a sufficient matrix rank to satisfy the condition, when there is antenna deficiency. Neither can a vertical expansion via oversampling, except when there is no overlapping among intra-user paths (a much stronger condition than the asynchrony condition). The **COD** strategy solves the antenna deficiency problem by combining vertical expansion with *temporal oversampling* and horizontal expansion with *spatial displacement*. Another unique advantage of **COD** is its multiplicity of eigenvalues which greatly facilitates the later signal recovery processing[5]. The paper first analyzes the theoretical footings for **COD** and follows with some illustrative simulation results in noisy channels.

1. INTRODUCTION

This paper considers the MIMO path separation problem at the receiving antenna array in up-link multipath propagation scenario. Our objective is to blindly separate different (both *inter-* and *intra-*user) paths by identifying their DOAs which will be subsequently used for source signal recovery. The technique is based on subspace approach which has become a dominant trend for DOA (Direction of Arrival) estimation, e.g. **MUSIC** [1], **TAM**[2] and **ESPRIT**[3].

For convenience, we normalize the symbol interval of the digital sources into unit one. Consider d digital users $s_i(t)$ $(1 \leq i \leq d)$ each transmitted through r_i independent multipaths with θ_{ij}, β_{ij}, $g_{ij}(t)$, τ_{ij} denoting the DOA, complex fading factor, overall temporal response and time delay for path (i,j) $(1 \leq i \leq d; 1 \leq j \leq r_i)$ respectively. Assume a ULA (uniformly-spaced linear antenna array) of M antenna elements at the receiver and denote $x_m(t)$ $(m = 1, \dots, M)$ as the baseband signal observed at m-th antenna element, then relating them with the parametric multipath model yields

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_M(t) \end{bmatrix}^T \quad (1)$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{r_i} \vec{\mathbf{a}}_{ij} \beta_{ij} g_{ij}(t - \tau_{ij}) * s_i(t) \quad (2)$$

This research was supported in part by a grant from Mitsubishi Electric Research Laboratories.

where $\vec{\mathbf{a}}_{ij} = \begin{bmatrix} 1 & e^{j2\pi\omega_{ij}} & \dots & e^{j2\pi(M-1)\omega_{ij}} \end{bmatrix}^T$ is well known as the *antenna response vector* ($\omega_{ij} = \frac{\Delta}{\lambda_o} sin(\theta_{ij})$ with Δ denoting the spacing of two adjacent antenna elements and λ_o the carrier wavelength). See [4, 5] for more details.

Most existing subspace processing techniques are based on a data matrix $\mathbf{X}$ formed from $x_m(t) (m = 1, \dots, M)$. For example, the $(M \times N)$ basic data matrix $\mathbf{X}$ adopted in many traditional approaches is

$$\mathbf{X} = \begin{bmatrix} \vec{\mathbf{x}}(0) & \vec{\mathbf{x}}(1) & \dots & \vec{\mathbf{x}}(N-1) \end{bmatrix} \quad (3)$$

For analysis of such data matrices, it is useful to define a Vandermonde *antenna response* matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} \vec{\mathbf{a}}_{11} & \dots & \vec{\mathbf{a}}_{1r_1} & \dots & \vec{\mathbf{a}}_{d1} & \dots & \vec{\mathbf{a}}_{dr_d} \end{bmatrix} \quad (4)$$

2. SEPARABILITY CONDITION

Definition 1 (Full Multipath Separability)

A data matrix $\mathbf{X}$ is said to have what we called a "full multipath separability property", if there exists a space-time factorization $\mathbf{X} = \mathbf{F_s} \cdot \mathbf{F_t}$ satisfying

1. *"$\mathbf{F_s}$-condition":*
   $\mathbf{F_s}$ has full column rank and each of its column vector reveals the spatial information concerning DOA of exactly one path. Moreover, each path can find at least one corresponding column vector in $\mathbf{F_s}$;
2. *"$\mathbf{F_t}$-condition":*
   $\mathbf{F_t}$ has full row rank and each of its row vector bears the temporal signal information from exactly the same path as that in the corresponding column of $\mathbf{F_s}$.

3. COD DIVERSITY COMPENSATION

Traditional approaches assume a large number of antenna elements ($M \gg r$) and the basic data matrix $\mathbf{X}$ in Eq[3] has a *space-time factorization* with $\mathbf{F_s} = \mathbf{A}$ satisfying the $\mathbf{F_s}$-*condition* in Definition 1. Under the much more challenging situation when M is small (possibly $M < r$), the $(M \times r)$ matrix $\mathbf{A}$ will no longer necessarily have full column rank. Therefore, it is necessary to (1) compensate the inadequacy in spatial diversity along vertical dimension of $\mathbf{A}$; (2) come up with a substituting space-time factorization which holds both $\mathbf{F_s}$ and $\mathbf{F_t}$-*condition* in Definition 1 and meanwhile (3) retain a proper structure of $\mathbf{F_s}$ for DOA information extraction. To this end, we propose the novel **COD** strategy which benefits from combined vertical expansion with *temporal oversampling* and horizontal expansion with *spatial displacement*.

3.1. Diversity Compensation Strategies

3.1.1. Temporal Oversampling

When a signal is sampled fractionally at the antenna, the *oversampling factor* P denotes the number of samples in one symbol interval. At p-th oversampling point, the data vector $\vec{\mathbf{x}}(t+\frac{p-1}{P})$ can still be obtained from Eq[2] by substituting τ_{ij} with $(\tau_{ij}-\frac{p-1}{P})$ and other parameters unchanged.

DEFINITION 2 (SHIFTED PATH DELAYS)
We define shifted time delay of path (i,j) at p-th oversampling point as $\tau_{ij}^{(p|P)}=(\tau_{ij}-\frac{p-1}{P})$, which can be expressed as the sum of shifted integer path delay $T_{ij}^{(p|P)}$ and fractional path delay $\gamma_{ij}^{(p|P)}$.

DEFINITION 3 (CONVOLUTIONAL VECTORS/ISI LENGTH)
We define the (individual-path) convolutional signal vector of time n as the sequence of source signals convolved at all the P oversampling points in path (i,j) within a symbol interval [n,n+1). More exactly,

$$\vec{\mathbf{s}}_{ij}(n)=\begin{bmatrix} s_i(-T_{ij}^{(P|P)}-1+n) \\ s_i(-T_{ij}^{(P|P)}-2+n) \\ \vdots \\ s_i(-T_{ij}^{(P|P)}-L_{ij}+n)\end{bmatrix} \tag{5}$$

where the ISI length L_{ij} is the number of (baud-rate) source signal samples convolved (it is determined primarily by the lengths of the transceiver FIR filters, the dispersive channel and in a minor way by $\gamma_{ij}^{(p|P)}$ $(1\le p\le P)$). The shifted convolutional temporal vector associated with oversampling point p is therefore

$$\vec{\mathbf{g}}_{ij}^{(p|P)}=\begin{bmatrix} g_{ij}(T_{ij}^{(P|P)}-T_{ij}^{(p|P)}+1-\gamma_{ij}^{(p|P)}) \\ \vdots \\ g_{ij}(T_{ij}^{(P|P)}-T_{ij}^{(p|P)}+L_{ij}-\gamma_{ij}^{(p|P)})\end{bmatrix}^T$$

Based on these, Eq[2] is equivalent to the vector form

$$\vec{\mathbf{x}}(n+\frac{p-1}{P})=\sum_{i=1}^{d}\sum_{j=1}^{r_i}\vec{\mathbf{a}}_{ij}\beta_{ij}\vec{\mathbf{g}}_{ij}^{(p|P)}\vec{\mathbf{s}}_{ij}(n) \tag{6}$$

3.1.2. Spatial Displacement

For diversity compensation we can form more than one virtual (spatial) sections from adjacent elements of a single antenna array. Here the *spatial displacement* K is defined as the total number of such sections (i.e. $\#[1,2,\ldots,(M-K+1)]$ form the first section, $\#[2,3,\ldots,(M-K+2)]$ form the second and so forth). The *partial antenna observation vector* associated with section k $(1\le k\le K)$ is denoted as

$$\vec{\mathbf{x}}^k(t)=\begin{bmatrix} x_k(t) & x_{k+1}(t) & \ldots & x_{M-K+k}(t)\end{bmatrix}^T$$

and the corresponding *partial antenna response* matrix $\mathbf{A}^{(k|K)}$ is the portion of k-th to $(M-K+k)$-th rows of $\mathbf{A}$.

Defining a $(r\times r)$ diagonal matrix $\mathbf{\Theta}=diag\{e^{j2\pi\omega_{ij}}\}$, then from the Vandermonde structure of $\mathbf{A}$ it is easy to verify the *shift-invariance property* of $\mathbf{A}^{(k|K)}$:

$$\mathbf{A}^{(k|K)}=\mathbf{A}^{(1|K)}\mathbf{\Theta}^{k-1}\quad(\forall k=1,\ldots,K) \tag{7}$$

3.1.3. Data Collection

We stack the antenna observations of section k at p-th oversampling point into a $((M-K+1)\times N)$ data matrix:

$$\mathbf{X}_{k|K}^{p|P}=\begin{bmatrix} \vec{\mathbf{x}}^k(\frac{p-1}{P}) & \vec{\mathbf{x}}^k(1+\frac{p-1}{P}) & \ldots & \vec{\mathbf{x}}^k(N-1+\frac{p-1}{P})\end{bmatrix}$$

LEMMA 1 (MATRIX FACTORIZATION)
Based on the parametric multipath channel model in Eq[2], $\mathbf{X}_{k|K}^{p|P}$ is equivalent to a product form

$$\mathbf{X}_{k|K}^{p|P}=\mathbf{A}^{(k|K)}\cdot\mathbf{G}^{(p|P)}\cdot\mathbf{S} \tag{8}$$

where $\mathbf{G}^{(p|P)}$ and $\mathbf{S}$ are $(r\times\sum_{i,j}L_{ij})$ and $(\sum_{i,j}L_{ij}\times N)$ matrices determined by $(p,\,g_{ij}(t))$ and $s_i(t)$ $(i=1,\ldots,d;j=1,\ldots,r_i)$ respectively together with other path parameters (e.g. β_{ij}, τ_{ij}).

Proof: Eq[8] can be proved if we transfer Eq[6] into a matrix form by setting the *shifted temporal response* matrix

$$\mathbf{G}^{(p|P)}=block-diag\{\beta_{ij}\vec{\mathbf{g}}_{ij}^{(p|P)}\} \tag{9}$$

and the *(expanded) source signal* matrix

$$\mathbf{S}=\begin{bmatrix} \mathbf{S}_1^T & \mathbf{S}_2^T & \ldots & \mathbf{S}_d^T\end{bmatrix}^T \tag{10}$$

$$\text{where } \mathbf{S}_i=\begin{bmatrix} \vec{\mathbf{s}}_{i1}(0) & \vec{\mathbf{s}}_{i1}(1) & \ldots & \vec{\mathbf{s}}_{i1}(N-1) \\ \vec{\mathbf{s}}_{i2}(0) & \vec{\mathbf{s}}_{i2}(1) & \ldots & \vec{\mathbf{s}}_{i2}(N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \vec{\mathbf{s}}_{ir_i}(0) & \vec{\mathbf{s}}_{ir_i}(1) & \ldots & \vec{\mathbf{s}}_{ir_i}(N-1)\end{bmatrix}.$$

3.2. COD Expansion and Space-Time Factorization

Combining *temporal oversampling* vertically and *spatial displacement* horizontally leads to the **COD** data matrix:

$$\mathbf{X}=\begin{bmatrix} \mathbf{X}_{1|K}^{1|P} & \mathbf{X}_{2|K}^{1|P} & \ldots & \mathbf{X}_{K|K}^{1|P} \\ \mathbf{X}_{1|K}^{2|P} & \mathbf{X}_{2|K}^{2|P} & \ldots & \mathbf{X}_{K|K}^{2|P} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{X}_{1|K}^{P|P} & \mathbf{X}_{2|K}^{P|P} & \ldots & \mathbf{X}_{K|K}^{P|P}\end{bmatrix} \tag{11}$$

THEOREM 1 (COD SPACE-TIME FACTORIZATION)
*For **COD** data matrix $\mathbf{X}$ in Eq[11], we have the space-time factorization with*

$$\mathbf{F_s}=\begin{bmatrix} \mathbf{A}^{(1|K)}\mathbf{G}^{(1|P)} \\ \mathbf{A}^{(1|K)}\mathbf{G}^{(2|P)} \\ \vdots \\ \mathbf{A}^{(1|K)}\mathbf{G}^{(P|P)}\end{bmatrix} \tag{12}$$

$$\mathbf{F_t}=\begin{bmatrix} \mathbf{S} & \mathbf{\Phi S} & \ldots & \mathbf{\Phi}^{K-1}\mathbf{S}\end{bmatrix} \tag{13}$$

where $\mathbf{\Phi}$ is a diagonal matrix with dimension $\sum_{i=1}^{d}\sum_{j=1}^{r_i}L_{ij}$.

Proof: Substitute each submatrix in $\mathbf{X}$ of Eq[11] with its matrix product form $\mathbf{X}_{k|K}^{p|P}=\mathbf{A}^{(k|K)}\cdot\mathbf{G}^{(p|P)}\cdot\mathbf{S}$ in [8]. Exploiting the *shift-invariance* of $\mathbf{A}^{(k|K)}$ in Eq[7] and block diagonal structure of $\mathbf{G}$, we have

$$\mathbf{X}_{k|K}^{p|P}=\mathbf{A}^{(1|K)}\cdot\mathbf{\Theta}^{k-1}\cdot\mathbf{G}^{(p|P)}\cdot\mathbf{S} \tag{14}$$

$$=\mathbf{A}^{(1|K)}\cdot\mathbf{G}^{(p|P)}\cdot\mathbf{\Phi}^{k-1}\cdot\mathbf{S} \tag{15}$$

where $\mathbf{\Phi}$ is the diagonal expansion of $\mathbf{\Theta}$ by repeating each of its element $(e^{j2\pi\omega_{ij}})$ L_{ij} times consecutively. Then it is trivial to obtain the space-time factorization in the lemma.

Theorem 2 (COD Separability Condition)
*Assuming independent users and P sufficiently large to yield a full column rank of $\mathbf{F_s}$, **COD** matrix $\mathbf{X}$ meets the "full multipath separability condition" if and only if*

$$K \geq max_{i=1}^{d}\{\frac{\sum_{j=1}^{r_i} L_{ij}}{L_i}\} \quad (16)$$

where L_i is defined as the rank of $\mathbf{S}_i$.

Proof: By inspection, each column(resp. row) of $\mathbf{F_s}(resp.\mathbf{F_t})$ bears the information of DOA (resp. signal and ISI) corresponding to exactly one path. Since P is assumed to be sufficiently large to yield the full column rank $Rank\{\mathbf{F_s}\} = \sum_{i,j} L_{i,j}$ (cf. Remark2), therefore, what remains to be verified is the full row rankness of $\mathbf{F_t}$. Assuming all users have different (and linear independent) signal sequences, it suffices that we verify the full-rankness of the rows attributed to each individual user. Each block column of $\mathbf{F_t}$ has rank $L_i \equiv Rank\{\mathbf{S}_i\}$ (attributed to the i-th user). Each increment of K brings in one additional column-block resulting in a net increase of rank by exactly L_i. (A disclaimer: we exclude pathological situations such as two DOA's happen to coincide, etc). The theorem is thus proved.

Remark 1 (Practical Range of K)
*Note that $L_i \leq \sum_{j=1}^{r_i} L_{ij}$ due to overlapping of convolution durations among intra-user paths. In most practical situations, it is reasonable to assume that all the paths have independent delays and channels so that the overlapping will not be very severe. Thus a displacement of $K = 2$ (or at most $K = 3$) suffices to meet Eq[16] in most cases. So we conclude that DOA estimation and path separation problem is theoretically tractable by **COD** as long as $M \geq 4(= K+1)$, i.e. it requires a very small antenna size.*

In some cases when the different intra-user paths arrive temporally far enough from each other such that $L_i = \sum_{j=1}^{r_i} L_{ij}$, then separability is already achieved by setting $K = 1$, i.e. no spatial displacement needed in Eq[11].

Remark 2 (Practical Range of P)
As a practical guideline, we suggest $P >> max_{i=1}^{d}\{\frac{\sum_{j=1}^{r_i} L_{ij}}{M}\}$ to guarantee the (numerical) full column rank of $\mathbf{F_s}$.

4. COD DOA FINDING ALGORITHM

With reference to Theorem 1, in **COD** expansion, each vertical block $\mathbf{A}^{(1|K)}\mathbf{G}^{(p|P)}$ of $\mathbf{F_s}$ retains the Vandermonde structure. Thus $\mathbf{X}$ possesses an "*intra-block shift-invariance property*", which is critical for DOA estimation. Let $\mathbf{Q_u}$ and $\mathbf{Q_l}$ be matrix- truncation operations on $\mathbf{F_s}$ which respectively extract the $(M-K)$ upper and $(M-K)$ lower rows out of each of the P vertical subblocks. We have

Theorem 3 (COD Eigenvalue Theorem)
*Given a **COD** data matrix $\mathbf{X}$ which satisfies the "full multipath separability property", apply SVD (Singular Value Decomposition) to obtain $\mathbf{X} = \mathbf{U\Sigma V}$. We assert that there exists an invertible matrix $\mathbf{R}$ such that*

$$(\mathbf{Q_u}[\mathbf{U}])^+ \cdot \mathbf{Q_l}[\mathbf{U}] = \mathbf{R}^{-1}\mathbf{\Phi R} \quad (17)$$

and hence DOAs of all the paths can be derived from the diagonal elements in $\mathbf{\Phi}$ ($(\cdot)^+$ denotes pseudo-inverse).

Proof: After the extractions, we have

$$\mathbf{Q_u F_s} = \begin{bmatrix} \mathbf{A}^{(1|K+1)}\mathbf{G}^{(1|P)} \\ \mathbf{A}^{(1|K+1)}\mathbf{G}^{(2|P)} \\ \vdots \\ \mathbf{A}^{(1|K+1)}\mathbf{G}^{(P|P)} \end{bmatrix} \quad \mathbf{Q_l F_s} = \begin{bmatrix} \mathbf{A}^{(2|K+1)}\bar{\mathbf{G}}^{(1|P)} \\ \mathbf{A}^{(2|K+1)}\bar{\mathbf{G}}^{(2|P)} \\ \vdots \\ \mathbf{A}^{(2|K+1)}\bar{\mathbf{G}}^{(P|P)} \end{bmatrix}$$

Applying Eq[7], it is immediate to show

$$\mathbf{Q_l}[\mathbf{F_s}] = (\mathbf{Q_u}[\mathbf{F_s}])\mathbf{\Phi} \quad (18)$$

Obviously, $\mathbf{U} = \mathbf{F_s R}$ for some nonsingular matrix $\mathbf{R}$ since the full column rankness of $\mathbf{F_s}$ implies that $\mathbf{U}$ and $\mathbf{F_s}$ share the same column span. It follows that

$$\begin{aligned} (\mathbf{Q_u}[\mathbf{U}])^+ \cdot \mathbf{Q_l}[\mathbf{U}] &= \mathbf{R}^{-1}(\mathbf{Q_u}[\mathbf{F_s}])^+ \cdot (\mathbf{Q_l}[\mathbf{F_s}])\mathbf{R} \\ &= \mathbf{R}^{-1}\mathbf{\Phi R} \quad (19) \end{aligned}$$

Eq[19] represents an eigenvalue decomposition where $\mathbf{\Phi}$ is the nonsingular, diagonal eigenvalue matrix (cf. the definition of $\mathbf{\Phi}$ in Section 3.2). The DOAs then can be calculated from its diagonal elements $e^{j2\pi\omega_{ij}}$ $(i = 1,\ldots,d; j = 1,\ldots,r_i)$ and each eigenvalue will have a multiplicity of L_{ij}.

The following **COD** algorithm can be regarded as a (block-)generalized version of **TAM** and **ESPRIT**[2, 3].

Algorithm 1 (COD DOA-Finding Algorithm)

1. *Select suitable parameters K and P according to Theorem 2, Remark 1,2, to form **COD** data matrix $\mathbf{X}$;*
2. *Apply SVD on $\mathbf{X}$ to obtain $\mathbf{X} = \mathbf{U\Sigma V}$;*
3. *Block-wise Extraction of $\mathbf{U}$ for $\mathbf{Q_u}[\mathbf{U}]$ and $\mathbf{Q_l}[\mathbf{U}]$;*
4. *Find the eigenvalues of $(\mathbf{Q_u}[\mathbf{U}])^+ \cdot \mathbf{Q_l}[\mathbf{U}]$;*
5. *Calculate DOAs θ_{ij} from the eigenvalues $e^{j2\pi\omega_{ij}}$.*

5. SIMULATION

The **COD** framework covers an extended family including basic **TAM/ESPRIT**[2, 3], **HT-TAM** (horizontal-temporally-oversampled TAM), **HS-TAM** (horizontal-spatially-displaced TAM), and combined **COD**. Our simulation study provides a comparison of their relative performances versus the maximum thermal SNR (among all the paths).

5.1. Performance Comparison When $M > r$

We have conducted 500 experiments with M = 10, r = 7, d=3, and the thermal SNR ranging from 0 to 20 db. In terms of finding *all* the paths, **COD** has a success rate around 80%-99% statistically, which is clearly superior to all the others (cf. Figure 1(a)). In terms of *all* paths DOA estimation accuracy, **COD** and **HT-TAM** deliver superior performance (around 1.0 to 0.3 degree in error) than the other two (cf. Figure 1(b)). From a practical perspective, the dominant paths would often suffice for the purpose of the signal recovery. As a selection criterion, we take advantage of the knowledge that the dominant path's eigenvalue are more likely to comply with unit-circle condition[5]. Similar as in *all* paths case, for *dominant* paths, **COD** has an extraction rate around 94% to 99% and estimation error around 0.75 to 0.2 degree, whose performance is well above those of others (cf. Figure 1(c)(d)). Figure 2 depicts the DOAs estimated by **TAM** and **COD**, when $M > r$. Note that **COD** successfully find all the paths (see the outermost ring) while **TAM** misses quite a few (the second ring).

5.2. COD Performance When $M \leq r$

When $M < r$, the traditional **TAM/ESPRIT** doesn't work. For investigation of **COD** performance, we have conducted 200 experiments for $M = 6$, $r = 10$, $d = 3$ with SNR ranging from 0 to 20 db. As shown in Figure 3, **COD** has an *all*-path-extraction rate of 60% to 90% with 1.6 to 0.45 degree error in DOA estimation. For *dominant* paths, the success rate is around 85% to 95% and it delivers an estimation accuracy of 1.1 to 0.35 degree in DOA error. Figure 4 depicts the DOAs estimated by **COD** when $M \leq r$.

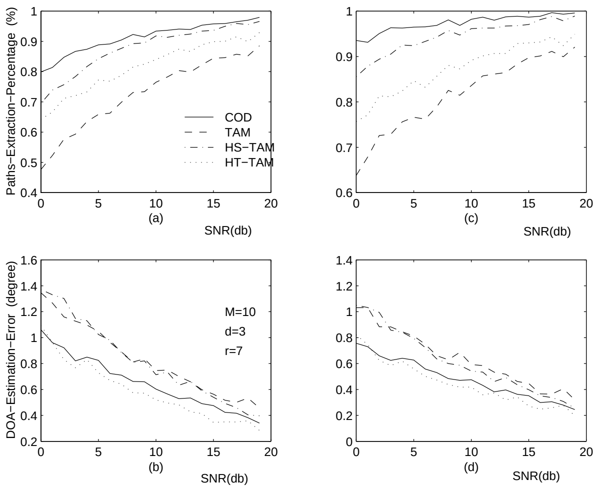

Fig. 1. Performance comparison for the case $M > r$: (a) all-paths extraction percentage; (b) all-paths DOA estimation error; (c) dominant-paths extraction percentage; (d) dominant-paths DOA estimation error vs. SNR.

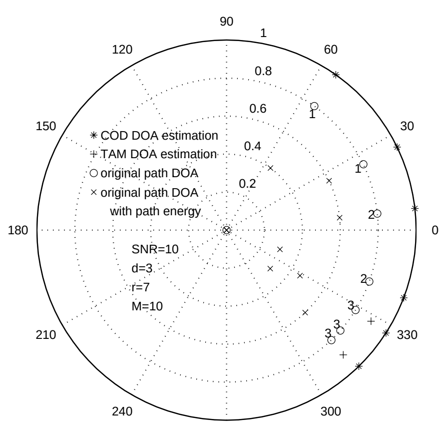

Fig. 2. DOAs estimated by TAM and COD, when $M > r$. Note COD finds all paths while TAM misses some.

6. REFERENCES

[1] R. O. Schmidt, "Multiple emitter location and Signal parameter estimation," *IEEE Trans. on Antenna and Propagation*, vol. AP-34, no. 3, 1986.

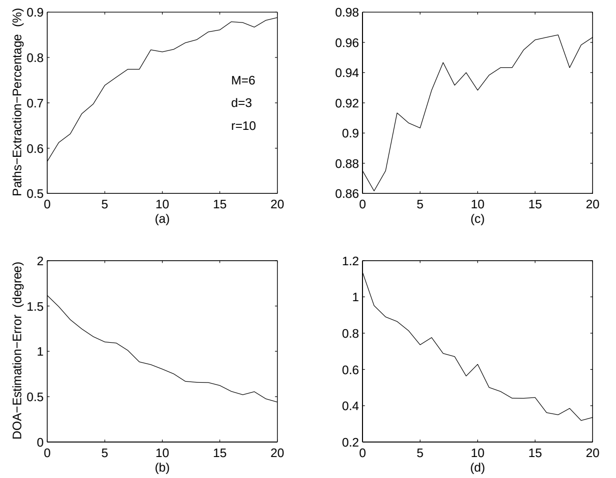

Fig. 3. COD performance when $M \leq r$: (a) all-paths extraction percentage; (b) all-paths DOA estimation error; (c) dominant-paths extraction percentage; (d) dominant-paths DOA estimation error vs. SNR.

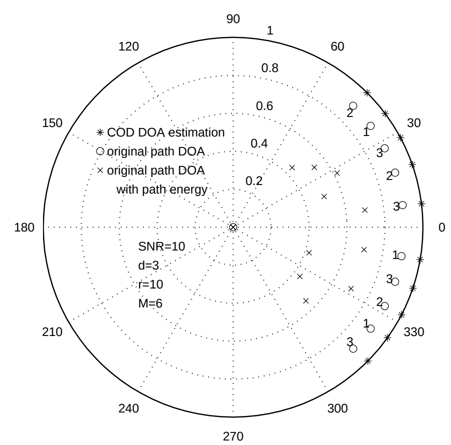

Fig. 4. DOAs as estimated by COD, when $M \leq r$. Note that TAM does not apply to situations when $M \leq r$.

[2] "State Space and Singular Value Decomposition Based on Approximation Methods for Harmonic Retrieval", *Journal of the Optical Society of America*, Dec. 1983. (S.Y. Kung, K.S. Arun, B. Rao)

[3] R. Roy and T. Kailath, "ESPRIT-estimation of signal parameters via rotational invariance techniques," *IEEE Trans. on ASSP*, vol. 37, July 1986.

[4] "Algebraic Method for Deterministic Blind Beamforming", Alle-Jan Van Der Veen, *Proceedings of the IEEE*, vol.86, No.10, October 1998

[5] "COD: Diversity-Adaptive Subspace Processing for Blind Multipath Separation and Signal Recovery", Xinying Zhang and S.Y. Kung, submitted to *Journal of VLSI Signal Processing Systems*, 2000.