H_∞ SMOOTHING FOR CONTINUOUS UNCERTAINS SYSTEMS

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ABSTRACT

Recently [1], a H_∞ smoother has been developed and it gives good results for noise uncertainties. Nevertheless, when appear uncertain parameters, its performances decrease significantly. That's why, in this paper, an estimator robust to noise uncertain properties and parameter uncertainties is presented. As in [1, 4, 6], the robust H_∞ smoother for uncertain systems is developed as a combination of two robust H_∞ filters. The robust performances, for both noise and parameter uncertainties, of this new approach are evaluated on a simple example.

1. INTRODUCTION

Optimal filtering problem have widely been treated over the last decades [5, 7, 9, 10]. Recently, problems involving noise uncertain statistics have been tackled. Various approaches have been developed in the continuous time domain [7, 10] and the discrete time domain as well [7, 10]. As noise statistics uncertainty can be considered as a model uncertainty, the aspect of model uncertainty has been of interest, recently. In the state space framework, these model uncertainties are related to the dynamic matrix and the output matrix of the system under consideration [3, 11]. In this situation, the uncertainty can be modelled as an exogenous noise [3] and the problem is handled as noise uncertain statistics problem. In the present paper, such an approach is adopted in order to derive an optimal H_∞ smoother robust to both noise uncertain statistics, dynamic matrix and output matrix uncertainties for continuous time systems represented by state space equations.

2. STATEMENT OF THE PROBLEM

Let us consider the following linear continuous time uncertain system:

$$\begin{aligned}\dot{x}(t) &= [A+\Delta A(t)]x(t)+B\,w(t)\\ y(t) &= [C+\Delta C(t)]x(t)+v(t)\\ z(t) &= L\,x(t) \ \ with\ t\in[0,T]\end{aligned} \qquad \text{Eq. 2.1}$$

where:

- ☒ $x\in\Re^n$, $y\in\Re^p$, $z\in\Re^m$, w and v are uncorrelated stationary zero mean white noises;
- ☒ A, B, C, L are constant matrices with suitable dimensions.

The following assumptions are made:

(A1). (A,C) detectable;

(A2). (A,B) controllable;

(A3). The noise covariance are defined as:

$$\begin{aligned}E[w(t).w'(\tau)] &= I\ \delta(t-\tau)\\ E[v(t).v'(\tau)] &= I\ \delta(t-\tau)\end{aligned} \qquad \text{Eq. 2.2}$$

with I denotes the identity matrix;

(A4). $\Delta A(t)$ and $\Delta C(t)$ represent the parameter uncertainties. They are modelled as:

$$\begin{bmatrix}\Delta A(t)\\ \Delta C(t)\end{bmatrix}=\begin{bmatrix}H_a(t)\\ H_c(t)\end{bmatrix}F(t)\,E(t) \qquad \text{Eq. 2.3}$$

with:

- ☒ $F(t)\in\Re^{i\times j}$ an unknown matrix with Lebesgue mesurable elements, satisfying $F'(t)\,F(t)\le I$;
- ☒ $H_a(t)$, $H_c(t)$ and $E(t)$ are known continuous bounded matrices with real values.

An H_∞ optimal smoother estimating the signal $z(t)$ from measurement $y(t)$ and for all acceptable uncertainties is sought considering the quadratic optimisation of the ratio between the energies of noises and estimation error respectively.

The estimation error is defined by:

$$e(t)=z(t)-\hat{z}(t) \qquad \text{Eq. 2.4}$$

Finally, the problem can be stated as follows:

Given γ, a strictly positive real, indicating the level of noise attenuation, a performance criterion is defined such as

$$J=\sup_{w,v\in L_2}\frac{\|e\|_2^2}{\|w\|_2^2+\|v\|_2^2}<\gamma^2 \qquad \text{Eq. 2.5}$$

Furthermore, the system is supposed to be quadratically stable; i.e., there exists a definite positive symetric matrix such that:

$$[A+\Delta A(t)]'P+\dot{P}+P[A+\Delta A(t)]<0 \qquad \text{Eq. 2.6}$$

<u>Remark 1:</u> In this paper, $\|v\|_2^2$ is the norm $L_2\,[0, T]$ and A' is the transpose matrix of A.

3. OPTIMAL H_∞ SMOOTHER

3.1. Definition of an auxiliary problem

In order to design an H_∞ smoother for uncertain continuous time systems, the combined use of a forward and a backward filter will be done. The construction of such a smoother is based on the result presented in [1]. According to [4], an optimal smoother can be obtained as the combination:

$$\hat{x}_l(t) = Q\left(Q_f^{-1}\hat{x}_f(t) + Q_b^{-1}\hat{x}_b(t)\right)$$
$$Q = \left(Q_f^{-1} + Q_b^{-1}\right)^{-1} \qquad \text{Eq. 3.1}$$

with $\hat{x}_f$ and $\hat{x}_b$ being the optimal estimates of the forward filter and the backward filter respectively.

These two filters are derived considering the following representation :

$$\dot{\eta}(t) = A\eta(t) + \begin{bmatrix} B & \frac{1}{\varepsilon}H_a \end{bmatrix}\overline{w}(t)$$
$$\quad with \quad \eta(0) = \eta_0 = 0 \qquad \text{Eq. 3.2}$$
$$y_m(t) = C\eta(t) + \begin{bmatrix} 0 & \frac{1}{\varepsilon}H_c \end{bmatrix}\overline{w}(t) + v(t)$$
$$z_m(t) = [L' \quad \varepsilon\gamma E']\eta(t) \quad with \quad t \in [0,T]$$

with $\eta \in \Re^n$ is the state, $z_m \in \Re^{m+j}$ the signal to restore, ε a stricly positive real number and:

$$\overline{B} = \begin{bmatrix} B & \frac{1}{\varepsilon}H_a \end{bmatrix} \quad \overline{D} = \begin{bmatrix} 0 & \frac{1}{\varepsilon}H_c \end{bmatrix}$$
$$\overline{w} = \begin{bmatrix} w \\ \varepsilon F E x \end{bmatrix} \quad \begin{matrix} \eta = x \\ y_m = y \end{matrix} \qquad \text{Eq. 3.3}$$

Furthermore, the initial conditions are assumed to be null.

An optimal estimate $\hat{z}_m$ of z_m is sought minimising the criterion : $J_m = \sup\limits_{\overline{w},v \in L_2} \dfrac{\|e_m\|_2^2}{\|\overline{w}\|_2^2 + \|v\|_2^2} < \gamma^2$ Eq. 3.4

with: $e_m = z_m - \hat{z}_m$

$$\hat{z}_m(t) = [\hat{z}' \quad 0]' = [L' \quad 0]'\hat{x}(t)$$

The H_∞ optimisation problem can be handled as a minimax one as follows :

$$J_m = \min_{e_m} \max_{\overline{w},v \in L_2} \left[\frac{1}{\gamma^2}\|e_m\|_2^2 - \|\overline{w}\|_2^2 - \|v\|_2^2\right] < 0 \qquad \text{Eq. 3.5}$$

This criterion can be rewritten as :

$$J_{fn} = J_m + V = \min_{\hat{x}_m} \max_{\overline{w},v \in L_2} \left[\frac{1}{\gamma^2}\|L(\eta - \hat{x}_l)\|_2^2 - \|\overline{w} - \overline{B}'P\eta\|_2^2 - \|v\|_2^2\right] < 0$$
Eq. 3.6

considering :

- the definite positive symmetric equation solution to the DRE (Differential Riccati Equation):

$$-\dot{P} = A'P + PA + P\overline{B}\,\overline{B}'P + \varepsilon^2 E'E \qquad \text{Eq. 3.7}$$

- the following Lyapunov function :

$$V = \int_0^T \frac{d}{dt}(\eta' P \eta)\,dt = 0$$
$$= -\varepsilon^2\|E\eta\|_2^2 - \|\overline{w} - \overline{B}'P\eta\|_2^2 + \|\overline{w}\|_2^2 \qquad \text{Eq. 3.8}$$

Assuming that : $\overline{w} = \hat{w} + \overline{B}'P\eta$

one can consider the auxiliary problem:

$$J'_m = \sup_{\hat{w},v \in L_2} \frac{\|L(\eta - \hat{x}_l)\|_2^2}{\|\hat{w}\|_2^2 + \|v\|_2^2} < \gamma^2 \qquad \text{Eq. 3.9}$$

with the associate model:

$$\begin{bmatrix} \dot{\eta} = \overline{A}\eta + \overline{B}\hat{w} \\ y_m = \overline{C}\eta + \overline{D}\hat{w} + v \end{bmatrix} \quad with \quad \begin{matrix} \overline{A} = A + \overline{B}\,\overline{B}'P \\ \overline{C} = C + \overline{D}\,\overline{B}'P \end{matrix} \qquad \text{Eq. 3.10}$$

Consequently, the H_∞ smoothing problem for uncertain systems can be considered as a classical H_∞ problem for noise uncertain statistics. Furthermore, the estimates $\hat{x}_f$ and $\hat{x}_b$ are derived considering the criteria:

$$J_f = \sup_{\hat{w},v \in L_2} \frac{\|L(\eta - \hat{x}_f)\|_2^2}{\|\hat{w}\|_2^2 + \|v\|_2^2} < \gamma_f^2 \qquad \text{Eq. 3.11}$$

$$J_b = \sup_{\hat{w},v \in L_2} \frac{\|L(\eta - \hat{x}_b)\|_2^2}{\|\hat{w}\|_2^2 + \|v\|_2^2} < \gamma_b^2 \qquad \text{Eq. 3.12}$$

γ must satisfy the two criteria defined by the equations (3.11) and (3.12). γ is defined as:

$$\gamma = \max(\gamma_f, \gamma_b) \quad with \quad \gamma, \gamma_f, \gamma_b > 0 \qquad \text{Eq. 3.13}$$

3.2. Forward Filter

The problem is to find a filter minimising J_f. The criterion can be written as a minimax criterion as follows:

$$\min_{\hat{x}_f} \max_{\hat{w},v \in L_2} J_f = \frac{1}{2}\left[\frac{1}{\gamma_f^2}\|L(\eta - \hat{x}_f)\|_2^2 - \|\hat{w}\|_2^2 - \|v\|_2^2\right]$$
Eq. 3.14

The optimal H_∞ filter minimising the criterion (3.11) is given by the following result:

<u>Theorem 1:</u> *Let us consider the linear continuous time - system defined by (2.1).The filter minimising (3.11) is given by:*

$$\dot{\hat{x}}_f = \overline{A}\hat{x}_f + \left(Q_f\overline{C}' + \overline{B}\overline{D}'\right)R(y_m - \overline{C}\hat{x}_f) \qquad \text{Eq. 3.15}$$

if and only if there exists a positive definite symmetric differentiable matrix $Q_f(t)$ satisfying:

$$\dot{Q}_f = \tilde{A}Q_f + Q_f\tilde{A}' - Q_f\left(\overline{C}'R\overline{C} - \frac{L'L}{\gamma_f^2}\right)Q_f + \overline{B}\,\Gamma\,\overline{B}'$$
$$\tilde{A} = \overline{A} - \overline{B}\Gamma\overline{D}'\overline{C} \quad and \quad R = \left(I + \overline{D}\,\overline{D}'\right)^{-1} \qquad \text{Eq. 3.16}$$
$$Q_f(0) = 0 \qquad \Gamma = \left(I + \overline{D}'\overline{D}\right)^{-1}$$

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Sketch of the proof:

Let the Hamiltonian:

$$H_f = \frac{1}{2}\left[(\eta - \hat{x}_f)'\frac{L'L}{\gamma_f^2}(\eta - \hat{x}_f) - \hat{w}'\hat{w} - (y_m - \overline{C}\eta - \overline{D}\hat{w})'(y_m - \overline{C}\eta - \overline{D}\hat{w})\right]$$
$$+ \Psi_f'\left[\overline{A}\eta + \overline{B}\hat{w}\right]$$
Eq. 3.17

One can easily derive the expression of the adjoint:

$$\Psi_f = -\frac{L'L}{\gamma_f^2}(\eta - \hat{x}_f) - \overline{C}'R(y_m - \overline{C}\eta) - \tilde{A}'\Psi_f \qquad \text{Eq. 3.18}$$
$$R = I - \overline{D}\Gamma\overline{D}' \quad and \quad \tilde{A} = \overline{A} - \overline{B}\Gamma\overline{D}'\overline{C}$$

using the Riccati transformation: $\eta = \hat{\eta}_f + Q_f \Psi_f$

where Q_f is a symmetric definite positive matrix, the following expression can be obtained after some calculation:

$$\dot{Q}_f = \tilde{A} Q_f + Q_f \tilde{A}' - Q_f \left(\bar{C}' R \bar{C} - \frac{L'L}{\gamma_f^2} \right) Q_f + \bar{B}\,\Gamma\,\bar{B}'$$

$$\dot{\hat{\eta}}_f = \bar{A} \hat{\eta}_f + \left(Q_f \bar{C}' + \overline{BD}' \right) R (y_m - \bar{C} \hat{\eta}_f) + Q_f \frac{L'L}{\gamma_f^2} \left(\hat{\eta}_f - \hat{x}_f \right)$$

Eq. 3.19

Considering the previous relation, the criterion can rewritten as:

$$\min_{\hat{x}_f} \max_{y_m \in L_2} J_f = \frac{1}{\gamma_f^2} \left\| L(\hat{\eta}_f - \hat{x}_f) \right\|_2^2 + \frac{1}{\gamma_f^2} \left\| L Q_f \Psi_f \right\|_2^2$$

$$- \left\| R^{\frac{1}{2}} \left(y_m - \bar{C} \hat{\eta}_f \right) \right\|_2^2 - \left\| \Gamma^{\frac{1}{2}} \bar{B}' \Psi_f \right\|_2^2 - \left\| R^{\frac{1}{2}} \bar{C} Q_f \Psi_f \right\|_2^2 \qquad \text{Eq. 3.20}$$

$$+ 2 \int_0^T \left[\left(y_m - \bar{C} \hat{\eta}_f \right)' R \bar{C} + (\hat{\eta}_f - \hat{x}_f)' \frac{L'L}{\gamma_f^2} \right] Q_f \Psi_f \, dt < 0$$

Considering the null term:

$$W_f = \int_0^T \frac{d}{dt} \left(\Psi_f' \, Q_f \, \Psi_f \right) dt = 0 \qquad \text{Eq. 3.21}$$

The criterion can be expressed as:

$$\min_{\hat{x}_f} \max_{y_m \in L_2} J_f = \frac{1}{\gamma_f^2} \left\| L(\hat{\eta}_f - \hat{x}_f) \right\|_2^2 - \left\| R^{\frac{1}{2}} \left(y_m - \bar{C} \hat{\eta}_f \right) \right\|_2^2 < 0$$

Eq. 3.22

In order to minimise this expression, the first term is equated to zero choosing $\hat{\eta}_f = \hat{x}_f$. This completes the proof. ♣♣♣

3.3. Backward filter

The estimation is done processing the data from T to 0. The optimal H∞ backward filter minimising criterion (3.12) is given by the following result:

Theorem 2: *Let us consider the linear system time invariant defined by (2.1). The a filter minimising (3.3) is given by the expressions:*

$$\frac{d}{d\tau} \hat{x}_b = -\bar{A}\,\hat{x}_b + \left(Q_b \bar{C}' - \bar{B}\,\bar{D}' \right) R \left(ym - \bar{C}\,\hat{x}_b \right) \qquad \text{Eq. 3.23}$$

if and only if there exists a positive definite symmetric matrix $Q_b(t)$, continue, differentiable and such as:

$$-\frac{d}{d\tau} Q_b = +\tilde{A} Q_b + Q_b \tilde{A}' + Q_b \left(\bar{C}' R \bar{C} - \frac{L'L}{\gamma_b^2} \right) Q_b - \bar{B}\,\Gamma\,\bar{B}'$$

$Q_b(T)=0$ Eq. 3.24

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The result can be prooved using the same technique as in the previous section. However, caution should be taken handling the backward differential equations arising in the optimisation.

3.4. H∞ optimal smoother

The synthesis of the smoother using the technique presented in section 3, gives rise the following result:

Theorem 3: *There is a H∞ optimal smoother such that $J < \gamma^2$, if there exists two H∞ optimal filters such that:*

Forward filter:

$$\dot{\hat{x}}_f = \bar{A} \hat{x}_f + \left(Q_f \bar{C}' + \overline{BD}' \right) R (y_m - \bar{C} \hat{x}_f)$$

$$\dot{Q}_f = \tilde{A} Q_f + Q_f \tilde{A}' - Q_f \left(\bar{C}' R \bar{C} - \frac{L'L}{\gamma^2} \right) Q_f + \bar{B}\,\Gamma\,\bar{B}' \qquad \text{Eq. 3.25}$$

Backward filter:

$$\frac{d}{d\tau} \hat{x}_b = -\bar{A}\,\hat{x}_b + \left(Q_b \bar{C}' - \bar{B}\,\bar{D}' \right) R \left(ym - \bar{C}\,\hat{x}_b \right)$$

$$-\frac{d}{d\tau} Q_b = \tilde{A} Q_b + Q_b \tilde{A}' + Q_b \left(\bar{C}' R \bar{C} - \frac{L'L}{\gamma^2} \right) Q_b - \bar{B}\,\Gamma\,\bar{B}'$$

Eq. 3.26

H∞ optimal smoother:

$$\hat{x}(t) = \left(Q_f^{-1} + Q_b^{-1} \right)^{-1} \left(Q_f^{-1} \hat{x}_f(t) + Q_b^{-1} \hat{x}_b(t) \right)$$

$$Q = Cov(x - \hat{x}) = \left(Q_f^{-1} + Q_b^{-1} \right)^{-1} \qquad \text{Eq. 3.27}$$

with Q_b *and* Q_b *defined has in Theorem 1 and 2.* ❑❑❑

4. APPLICATION

Let us consider the nominal expression for linear continuous time system:

$$\dot{x}(t) = A\,x(t) + B\,w(t)$$

$$y(t) = C\,x(t) + +D\,w(t) + v(t) \qquad \text{Eq. 4.1}$$

$$z(t) = L\,x(t) \text{ with } t \in [0,T]$$

$$A = diag\{-1,-2,-3\} \qquad B = 25[1 \;\; 1 \;\; -1]$$

$$C = [-1 \;\; 2 \;\; 1] \qquad L = [1 \;\; 1 \;\; 1] \text{ and } D = 1$$

As no H∞ smoother for both uncertain noise statistics and plant parameters do exist, the performance of the new tool will be compared to the H_2 smoother. The evaluation of the H∞ smoother will be processed in two steps.

Firstly, in order to show the robustness regarding uncertain measurement noise statistics, a study will be done assuming no model parameter uncertainty. In the example, both H_2 and H∞ are designed with the same nominal value of the measurement noise statistics. However, the real value is evolving in a ratio of 1/5 and 5 with regard to the nominal value. Figure 1 shows the efficiency of the H∞ smoother.

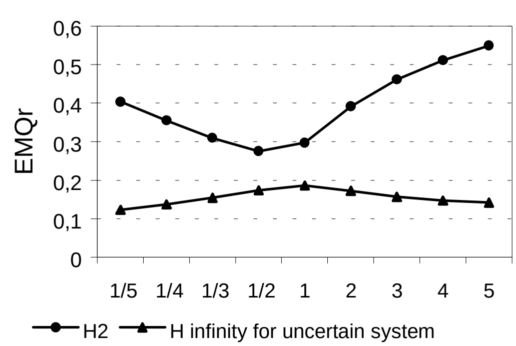

FIG. 1: Robust H_∞ smoother, and H_2 smoother. Variation of noise properties between 1/5 to 5 of the nominal value.

Secondly, as the H_∞ smoother has been developed for uncertain plant parameters, parametric robustness is studied considering no noise uncertainty.

Consequently, the following parameter variation is considered:

$$A = diag\{-1, -2 + \delta, -3\} \text{ with } \delta = \text{-0.2 to 0.2}$$

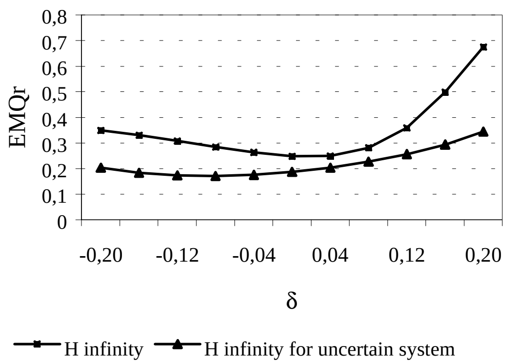

FIG. 2: Robust H_∞ smoother, and H_∞ smoother.

The figure 2 shows the robustness property of the H_∞ smoother with respect to the uncertain parameter.

5. CONCLUSION

The incertitude issue of modelisation has been extended to modelisation of noise as well as for system. Since today, for the non causal estimation topic for linear continuous time systems, we found in litterature a H_∞ smoother [1] respect to noise statistical properties. The problem is that one doesn't give good performance when we're confronted to uncertain parameters. So a robust H_∞ smoother has been developed in this paper in order to fill this lack and offer good performance regarding to modelisation uncertainties. The simulation has shown the performance increase that offers such a technique. When uncertainties disappear, smoothers performances tend to join. The H_∞ smoother appeared to be a sub-optimal version a sub-optimal version of the robust H_∞ smoother defined in this paper.

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