CONVERGENCE ANALYSIS OF A NEW BLIND EQUALIZATION ALGORITHM WITH M-ARY PSK CHANNEL INPUTS

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ABSTRACT

In our previously proposed equalization criterion with M-PSK modulated channel inputs [10], perfect equalization was guaranteed in principle under constrained optimization. The algorithm was implemented based on stochastic gradient method. In this paper, we investigate the convergence property of that algorithm. By carefully examining all stationary points of the objective function, we prove that all other points are unstable except the desired solution. The effects of the ary number M and the background noise on the performance of the proposed equalizer are studied in detail in our simulations. Satisfactory bit error rate performance and output constellation are observed.

1. INTRODUCTION

In wireless digital communications, signals are usually modulated before transmission. Due to multipath propagation, channel equalization is required to suppress inter-symbol interference (ISI). When the modulated signals have constant modulus property, the constant modulus algorithm (CMA) is one of those very efficient methods for equalization. It is known that CMA was proposed by [2], [6] and developed independently by [8]. New criteria were derived and conditions for blind deconvolutions were generally analyzed in [7]. A detailed survey of this algorithm is recently presented by [4].

All those contributions employ only the amplitude information of modulated signals. However, modulated signals possess properties not only on their moduli, but also on their phases. For example, phase shift keying (PSK) modulated signals uniformly distribute on a circle, pulse amplitude modulation (PAM) signals equally space on the x-axis. These information is usually known at the receiver. Their exploitation helps channel equalization, as has been shown in constellation matching based approach [5]. Initial efforts have also been made in [9], [10] where statistical properties of PAM or M-PSK modulated signals are considered in the equalization cost/objective function.

In [10], it was found that the k-th order moment of a M-PSK signal is non-zero only when k is a multiple of M. In such a case the k-th order moment of the equalized signal becomes a constant in the absence of noise. Such constant is a function of composite channel parameters including the effects of both the communication channel and the equalizer. Similarly, the equalizer's output power is also parameterized by composite channel parameters. Perfect equalization (PE) requires that the composite channel have single impulse response and was achieved in [10] by maximizing the absolute value of the M-th order moment of the equalized signal under the output power constraint. The equalizer was recursively updated by stochastic gradient method. Simulation results showed that the equalizer had a better performance than those conventional CMA algorithms. For such implementation, one may ask if the algorithm can converge to its globally optimal point.

In this paper, convergence property of that algorithm is analyzed. We first express the objective function used in [10] as a function of composite channel parameters, and obtain a set of stationary points. Among those, we further prove that the optimal point is neither local maximum nor local minimum. Therefore it is concluded that all are unstable points except the desired one. Comparisons with [2], [7] were made in [10], and thus omitted in the current paper. However, simulations are conducted to show the applicability once more. Moreover, more new results are provided to show the effects of the ary number M and the background noise on the equalizer's performance in terms of the output constellation and bit error rate (BER).

2. PROBLEM STATEMENT

Consider blind equalization problem with M-PSK inputs $\boldsymbol{s}(n)$ and channel output $\boldsymbol{x}(n)$ [3]

$$\boldsymbol{x}(n) = \boldsymbol{H}\boldsymbol{s}(n), \;\; y_n = \boldsymbol{f}^H\boldsymbol{x}(n) = \boldsymbol{a}^T\boldsymbol{s}(n) \qquad (1)$$

where $\boldsymbol{H}$ is the channel matrix, y_n is the output of the equalizer $\boldsymbol{f}$, $\boldsymbol{a}^T = \boldsymbol{f}^H\boldsymbol{H}$ is the combined response of the channel and the equalizer. Perfect equalization can be achieved if $\boldsymbol{a}$ has only one non-zero element [7]

$$\boldsymbol{a} = e^{j\theta}[0, \cdots, 0, 1, 0, \cdots, 0]^T \qquad (2)$$

Different criteria can be used to obtain the equalizer. In [10], we proposed a new criterion to obtain the equalizer with M-ary PSK inputs

$$\max |E\{y_n^M\}|^2, \quad \text{subject to } E\{|y_n|^2\} = 1 \qquad (3)$$

Lagrange cost function was constructed to solve this constrained maximization problem

$$J_1(\boldsymbol{f}) = E\{(\boldsymbol{f}^H\boldsymbol{x})^M\}E\{(\boldsymbol{x}^H\boldsymbol{f})^M\} + \lambda(\boldsymbol{f}^H\boldsymbol{R}\boldsymbol{f} - 1) \quad (4)$$

where $\boldsymbol{R} = E\{\boldsymbol{x}\boldsymbol{x}^H\}$, unknown multiplier λ was obtained from power constraint. Finally, the algorithm was implemented by gradient ascent method

$$\boldsymbol{f} = \boldsymbol{f} + \mu M b^*[E\{(\boldsymbol{f}^H\boldsymbol{x})^{M-1}\boldsymbol{x}\} - b\boldsymbol{R}\boldsymbol{f}] \qquad (5)$$

with step size μ and defined constant $b = E\{(\boldsymbol{f}^H\boldsymbol{x})^M\}$. Since the new criterion considers the phase characteristics of the M-PSK signals, which are not considered in CMA algorithm, a significant performance gain was achieved [10] in terms of ISI and decoding error. However, the convergence property of the new algorithm has not been investigated. In this paper, we will prove that the equalizer converges to the only global maximum that is the desirable response.

3. CONVERGENCE ANALYSIS

Our algorithm is similar to [7] in the sense that both are based on constrained maximization. Some ideas therein will be borrowed in the analysis. Perfect equalization should satisfy the condition (2). It requires combined channel parameters a_l have only one non-zero element. The analysis will proceed by examining all stationary points, and showing no possible local maximum and local minimum in the solution set for the combined channel parameters.

According to the criterion, equalizer $\boldsymbol{f}$ is obtained from optimization statement in (3). By using the result of [10] $E\{y_n^M\} = \sum_l a_l^M$ and $E\{|y_n|^2\} = \sum |a_l|^2$, the equalization criterion is thus equivalent to the following constrained maximization with respect to combined channel parameters a_l:

$$\max J(\boldsymbol{a}) = |\sum_l a_l^M|^2, \quad \text{subject to } \sum_l |a_l|^2 = 1 \qquad (6)$$

Solutions to (6) form a set of stationary points, to which the equalizer may converge. To obtain the solution, we construct the following Lagrange function:

$$J(\boldsymbol{a}, \lambda_1) = |\sum_l a_l^M|^2 + \lambda_1(\sum_l |a_l|^2 - 1) \qquad (7)$$

where λ_1 is a Lagrange multiplier. At stationary points, $\boldsymbol{a}$ should satisfy the following partial derivative equations:

$$\frac{\partial J(\boldsymbol{a}, \lambda_1)}{\partial a_l} = M a_l^{M-1} \sum_k (a_k^*)^M + \lambda_1 a_l^* = 0 \qquad (8)$$

λ_1 is obtained by multiplying both sides of (8) by a_l and summing results for all l under constraint $\sum_l |a_l|^2 = 1$

$$-M \sum_l (a_l^*)^M \sum_k a_k^M = \lambda_1 \sum_l |a_l|^2 = \lambda_1 \qquad (9)$$

Since $\lambda_1 \neq 0$, $\sum_k (a_k^*)^M \neq 0$. Thus after substituting λ_1 into (8), we obtain

$$a_l^{M-1} - a_l^* \sum_k a_k^M = 0 \qquad (10)$$

If we define $a_l \triangleq r_l e^{j\theta_l}$, we get a set of stationary points with zeros and non-zero terms: $r_l = 0 \Longleftrightarrow a_l = 0$ and $r_l^{M-2} e^{jM\theta_l} = \sum_k a_k^M = constant, \quad \forall l$. Therefore the nonzero terms in a single stationary point satisfy

$$\begin{cases} r_l = r_m \text{ where } l \neq m \\ M\theta_l = M\theta_m + 2k\pi \quad k \text{ is any integer} \end{cases} \qquad (11)$$

If there are totally N nonzero components in $\boldsymbol{a}$, then combining (11) and the constraint $\sum_l |a_l|^2 = 1$, we further get $r_l^2 = \frac{1}{N}$ for non-zero elements. For convenience and from now on, we denote the solutions to (8) under the constraint for a specified N as vectors $\boldsymbol{a}^N = [a_1^N \; a_2^N \ldots]^T$ such that each element a_l^N satisfies[1]

$$a_l^N \triangleq r_l^N e^{j\theta_l^N} = \begin{cases} \frac{1}{\sqrt{N}} e^{j\theta_l^N} & l \in I_N \\ 0 & \text{otherwise} \end{cases} \qquad (12)$$

where I_N is any N-element subset of integers. The phase of the nonzero terms should satisfy (11). Therefore, the set of stationary points for (7) has elements $\boldsymbol{a}^N$, $N = 1, 2, 3, \cdots$. By *Theorem 1* in [10], $\boldsymbol{a}^1$ is the set of global maximum. Next we will show that all other stationary points $\boldsymbol{a}^N$, $N = 2, 3, \cdots$ are unstable.

3.1. Local maximum?

For any prespecified N, $N \geq 2$, we define a set $B(\boldsymbol{a})$ whose elements satisfy the constraint with certain phase relations

$$B(\boldsymbol{a}) = \{ \; \sum_l r_l^2 = 1, M\theta_l = M\theta_m + 2k\pi \text{ for } l, m \in I_N, \; k \text{ is any integer}, \; r_l = 0 \text{ otherwise} \} \qquad (13)$$

Clearly, $\boldsymbol{a}^N \in B(\boldsymbol{a}^N) \subseteq B(\boldsymbol{a})$. To show that $\boldsymbol{a}^N$, $N = 2, 3, \ldots$ are not local maximum, we will prove $J(\boldsymbol{a}) \geq J(\boldsymbol{a}^N)$, where $\boldsymbol{a} \in B(\boldsymbol{a})$. First we define $\boldsymbol{\alpha}$ as $\boldsymbol{\alpha} = [\alpha_1 \; \alpha_2 \cdots]^T$ whose element is $\alpha_l = |a_l|^2 - |a_l^N|^2 = r_l^2 - \frac{1}{N}$. Under this definition, α_l satisfies

$$\sum_l \alpha_l = 0, \; \begin{cases} -\frac{1}{N} \leq \alpha_l \leq (1 - \frac{1}{N}) & \text{for } l \in I_N \\ \alpha_l = 0 & \text{otherwise} \end{cases} \qquad (14)$$

[1] Without confusing from the context, notation with superscript N is used to represent such set instead of the power of the argument.

Then, we can evaluate the values of objective function $J(\boldsymbol{a})$ at $\boldsymbol{a}^N$ and $\boldsymbol{a}$ respectively, and find their relative relation.

First from (13), we have $e^{jM\theta_l^N} = e^{jM\theta_m^N}$. Then

$$\sqrt{J(\boldsymbol{a}^N)} = |\sum_l (a_l^N)^M| = |\sum_l (r_l^N)^M| = \frac{1}{N^{L-1}} \quad (15)$$

where $L = \frac{M}{2}$. Since $\boldsymbol{a}$ is $\in B(\boldsymbol{a})$ in (13), we similarly obtain

$$\sqrt{J(\boldsymbol{a})} = |\sum_l r_l^M e^{jM\theta_l}| = \sum_{l\in I_N} (\frac{1}{N} + \alpha_l)^L \quad (16)$$

Next, we compare (16) with (15) under constraints (14). Construct a new constrained function

$$E(\boldsymbol{\alpha}) = \sum_{l\in I_N} (\frac{1}{N} + \alpha_l)^L - \frac{1}{N^{L-1}} - \lambda_2(\sum_{l\in I_N} \alpha_l) \quad (17)$$

then

$$\frac{\partial E(\boldsymbol{\alpha})}{\partial \alpha_l} = L(\frac{1}{N} + \alpha_l)^{L-1} - \lambda_2 = 0 \quad (18)$$

From (18) and the constraint (14), we can conclude that at stationary points of $E(\boldsymbol{\alpha})$, all non-zero α_l should be equal. Thus the only stationary point of $E(\boldsymbol{\alpha})$ is a zero vector $\boldsymbol{\alpha}_{opt} = [0, \cdots, 0]^T$.

To see if this point is the only minimum point, we continue to evaluate the Hessian of $E(\boldsymbol{\alpha})$. According to (18), the Hessian matrix is seen to be diagonal. Its determinant can be computed

$$|\boldsymbol{D}_N| = \prod_l \frac{\partial^2 E}{\partial \alpha_l^2} = \prod_l L(L-1)(\frac{1}{N} + \alpha_l)^{L-2} \quad (19)$$

Clearly $|\boldsymbol{D}_N|_{\boldsymbol{\alpha}_{opt}} > 0$. Therefore $\boldsymbol{\alpha}_{opt} = [0, \cdots, 0]^T$ is the only minimum point of $E(\boldsymbol{\alpha})$ [1] which means that $E(\boldsymbol{\alpha}) \geq E(\boldsymbol{\alpha}_{opt}) = 0$. Under the constraint (14), it is equivalent to

$$\sum_{l\in I_N} (\frac{1}{N} + \alpha_l)^L \geq \frac{1}{N^{L-1}}$$

Therefore

$$J(\boldsymbol{a}) \geq J(\boldsymbol{a}^N)$$

The inequality is strict when $\alpha_l \neq 0$ for some $l \in I_N$.

Up to now, we have proved $\boldsymbol{a}^N$, $N = 2, 3, ...$ can not be a local maximum of $J(\boldsymbol{a})$ over $B(\boldsymbol{a})$. Next, we will further investigate if $\boldsymbol{a}^N$, $N = 2, 3, ...$ can be a local minimum.

3.2. Local minimum?

Similar to Shalvi's approach, we construct another set $\widetilde{\boldsymbol{a}} = [\widetilde{a}_1\ \widetilde{a}_2 \cdots]^T = [r_1 e^{j\theta_1}\ r_2 e^{j\theta_2} \cdots]^T$ with its amplitude and phase satisfying

$$r_l = \begin{cases} \sqrt{1-\epsilon}\, r_l^N & l \in I_N \\ \sqrt{\epsilon} & l = \widetilde{p} \\ 0 & \text{otherwise} \end{cases} \quad (20)$$

$$M\theta_l = M\theta_m + 2k\pi \ \text{ where } k \text{ is any integer} \quad (21)$$

where $0 < \epsilon < \frac{1}{N+1}$ and $\widetilde{p}$ is any integer such that $\widetilde{p}$ does not belong to I_N. Obviously $\widetilde{\boldsymbol{a}} \in B(\boldsymbol{a})$ and $\sum_l |\widetilde{a}_l|^2 = 1$. Now, we will prove $J(\widetilde{\boldsymbol{a}}) < J(\boldsymbol{a}^N)$ by first showing that $|\sum_l \widetilde{a}_l^M| < |\sum_l (a_l^N)^M|$. By direct calculation, we get

$$\begin{aligned} |\sum_l \widetilde{a}_l^M| &= |\sum_{l\in I_N} (\sqrt{1-\epsilon} s_l^N)^M + \epsilon^L e^{jM\theta_{\widetilde{p}}}| \\ &= |\sum_{l\in I_N} (1-\epsilon)^L (r_l^N)^M e^{jM\theta_l} + \epsilon^L e^{jM\theta_{\widetilde{p}}}| \\ &= (1-\epsilon)^L \frac{1}{N^{L-1}} + \epsilon^L \end{aligned} \quad (22)$$

Then we can construct function $G(\epsilon)$

$$\begin{aligned} G(\epsilon) &= |\sum_l \widetilde{a}_l^M| - |\sum_l (a_l^N)^M| \\ &= (1-\epsilon)^L \frac{1}{N^{L-1}} + \epsilon^L - \frac{1}{N^{L-1}} \end{aligned} \quad (23)$$

The stationary point of $G(\epsilon)$ satisfies

$$\frac{\partial G(\epsilon)}{\partial \epsilon} = -L(1-\epsilon)^{L-1} \frac{1}{N^{L-1}} + L\epsilon^{L-1} = 0 \quad (24)$$

which gives $L-1$ repetitive roots $\epsilon = \frac{1}{N+1}$. There are no other stationary points since $G(\epsilon)$ is a polynomial of order L. The second derivative of $G(\epsilon)$ has the form

$$\frac{\partial^2 G}{\partial \epsilon^2} = L(L-1)(1-\epsilon)^{L-2} \frac{1}{N^{L-1}} + L(L-1)\epsilon^{L-2} \quad (25)$$

Clearly, $\frac{\partial^2 G}{\partial \epsilon^2}|_{\epsilon = \frac{1}{N+1}} > 0$. Therefore, $\epsilon = \frac{1}{N+1}$ is the only minimum of $G(\epsilon)$. Observing $G(\epsilon = 0) = 0$ and

$$G(\epsilon = \frac{1}{N+1}) = \frac{1}{(N+1)^{L-1}} - \frac{1}{N^{L-1}} < 0$$

we conclude that, when $0 < \epsilon < \frac{1}{N+1}$, $G(\epsilon) < 0$ which means $|\sum_l \widetilde{a}_l^M| < |\sum_l (a_l^N)^M|$. Hence $J(\widetilde{\boldsymbol{a}}) < J(\boldsymbol{a}^N)$. This shows that $\boldsymbol{a}^N$, $N = 2, 3...$, can not be a local minimum.

Combining the previous two results, it is established that $\boldsymbol{a}^N$, $N = 2, 3, ...$ must be a saddle point. Therefore, $\boldsymbol{a}^1$ is the unique globally optimal solution. Perfect equalization is then achieved in the absence of noise, as also verified by our simulation results.

4. NUMERICAL RESULTS

Since performance comparison between the proposed algorithm and conventional CMA was shown in [10], we only present results for our method with M-PSK inputs. We consider a non-minimum phase channel used in [7], and use 12-tap equalizer which is initialized as a unitary vector with

"1" in the middle. The step size is set to be $\mu = \frac{0.02}{M}$. Fig.1 is the output diagram with 8-ary PSK inputs. As expected, the equalizer's output converges to eight desirable constellations accurately, indicating perfect equalization. We also test the effect of M and background noise on the BER. For different M, no detection errors are observed after the algorithm converges. Thus the average BERs from 100 realizations are compared in Fig. 2 from the beginning of iteration. It is seen that the BER decreases with M. Fig. 3 shows the average BERs from 1000 realizations for $M = 4, 8, 16, 32$ in the presence of different additive white Gaussian noise (AWGN). SNR varies from $0dB$ to $32dB$ with $2dB$ increment. This result further confirms the convergence of our new algorithm in a noisy environment.

5. REFERENCES

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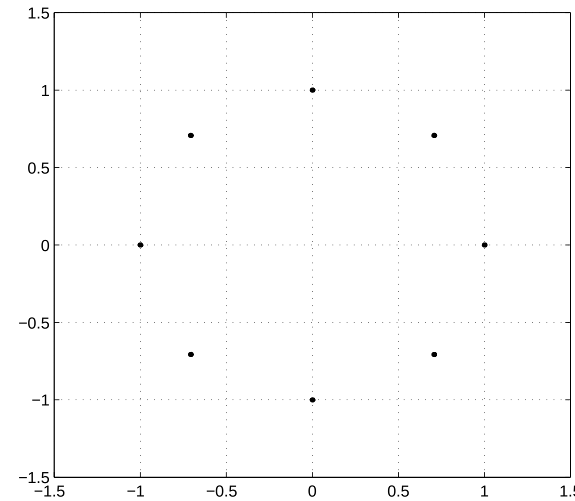

Fig. 1. Output constellation of the proposed algorithm for 8-PSK modulation.

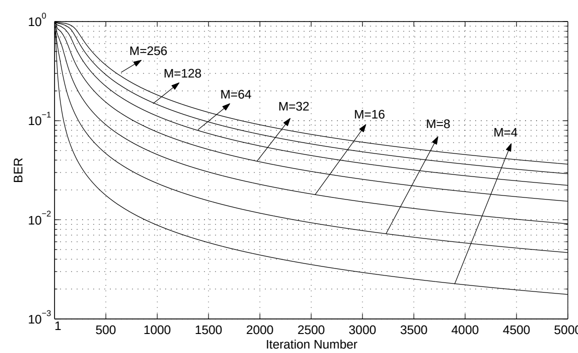

Fig. 2. Effect of the ary number M on the equalization performance.

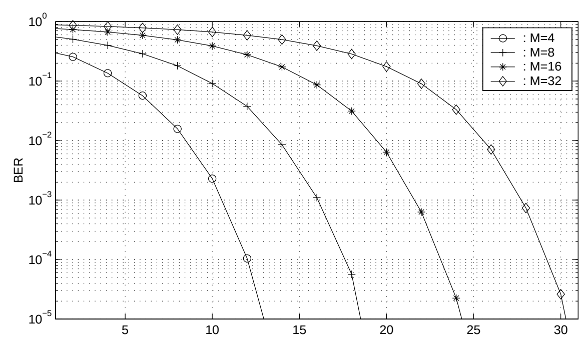

Fig. 3. Performance of the equalizer with different AWGN and different M.