Alperts Multi-wavelets From Spline Super-Functions

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ABSTRACT

For multi-wavelets generalized left eigenvectors of the matrix H_f a finite portion of down-sampled convolution matrix H determine the combinations of scaling functions that produce the desired spline or scaling function from which polynomials of desired degree can be reproduced. This condition is used to construct Alpert's multi-wavelets with multiplicity two, three and four and with approximation orders two, three and four respectively. Higher multiplicity Alpert multi-wavelets can also be constructed using this new method.

1. INTRODUCTION

Recently, multi-wavelets (wavelets with more than one scaling and wavelet functions) have been the focus of a lot of research in signal processing and pure mathematics [1-11]. The interest in multi-wavelets is mainly due to the fact that they produce promising results in many applications such as image compression and denoising [11]. Their success stems from the fact that they can simultaneously posses the good properties of orthogonality, symmetry, high approximation order and short support. The first multi-wavelet construction is due to Alpert [1]. In [1] a new bases for L^2 is constructed using r scaling functions with approximation order $p=r$. The aim in [1] was to obtain a basis which results in sparse representation for several kind of integral equations. The method in [1] relies on the Gram-Schmidt orthogonalization procedure. In this work Alpert's multi-wavelets are constructed using an alternative method. Here approximation order requirement is formulated in a matrix form that recognizes the generalized left eigenvectors of a portion of down-sampled convolution matrix H as the coefficients that enter the linear combination of scaling functions which produces the desired spline. This condition is then used together with orthogonality requirement to construct Alpert like multi-scaling functions with short support and high approximation order. The paper is organized as follows. In section II basic definitions of multi-wavelets and approximation order are made. For the multi-wavelet case a matrix equation is formulated for the desired approximation order based on the super function idea. In section III Alpert multi-scaling functions with multiplicity two, three and four are constructed. Conclusions are presented in section IV.

2. MULTI-WAVELETS AND APPROXIMATION ORDER

A multi-wavelet basis is characterized by r scaling and r wavelet functions. Here r denotes the multiplicity in the vector setting with $r > 1$. Multi scaling functions satisfy the matrix dilation equation or the refinement equation

$$\Phi(t) = \sum_k c_k \Phi(2t-k) \tag{1}$$

Similarly for the multi-wavelets the matrix dilation equation is given by

$$\Psi(t) = \sum_k d_k \Phi(2t-k) \tag{2}$$

where $\Phi(t)=[\phi_0(t)\ \phi_1(t)\ldots\phi_{r-1}(t)]^T$, $\Psi(t)=[\psi_0(t)\ \psi_1(t)..\psi_{r-1}(t)]^T$ and c_k and d_k are r by r real matrices that iteratively and with rescaling define the scaling and wavelet functions respectively. The support of the multi-scaling or wavelet functions depends on the length of the multi-filter coefficients.

Multi-wavelets are said to have approximation order p if a linear combination of the scaling functions can reproduce polynomials up to degree p-1. Then, the polynomials up to degree p-1 are in the linear span of the scaling space spanned by the shifts of scaling functions.

In order to achieve the desired approximation order in the multi-wavelet context, it is proposed that a finite linear combination of scaling functions will be able to reproduce the super function with the desired approximation order, i.e.,

$$\sum_n \sum_k a_k^n \cdot \phi_n(t-k) = \sum_l h(l) \left\{ \sum_n \sum_k a_k^n \cdot \phi_n(2t-k-l) \right\} \tag{3}$$

where $h[l]$ is the sequence that defines the spline and a_k^n is the sequence that enter the finite linear combination. The above equation has the exact form of a scalar dilation equation.

The left hand side is the scaling function $f(t)$ at one scale and the right hand side is the scaling function at twice the scale. Here

$$f(t) = \sum_n \sum_k a_k^n \cdot \phi_n(t-k) \tag{4}$$

The linear combination must produce the spline f(t) in the interval where it is supported and must give zero outside that interval.

Expanding the left hand side of (3) using the matrix dilation equation and matching terms using the right hand side results in the following matrix equation for the desired approximation order:

$$H_f^T \mathbf{x} = B_f^T \mathbf{x} \tag{5}$$

where $x_k=[a_k^0\quad a_k^1 \ldots\ldots a_k^{r-1}]^T$, k=0,1,2,…,K-1, I is the r by r identity matrix and H_f and B_f are finite portions of the infinite matrices H and B respectively and H and B

$$H=\begin{pmatrix} \ddots & & & & & & & & \\ & c_0 & c_1 & c_2 & \cdot & \cdot & c_{N-1} & & \\ & & & c_0 & c_1 & c_2 & \cdot & \cdot & c_{N-1} \\ & & & & & & & & \ddots \end{pmatrix} \quad (6)$$

$$B=\begin{pmatrix} \ddots & & & & & & & \\ & h[0]I & h[1]I & \cdot & \cdot & h[L-1]I & & \\ & & h[0]I & h[1]I & \cdot & \cdot & h[L-1]I & \\ & & & & & & & \ddots \end{pmatrix} \quad (7)$$

In (5), only a finite portion of matrices H and B, relevant for the desired length of scaling functions and approximation order, is considered. Equation (5) is exactly the generalized eigenvalue problem. Generalized left eigenvectors x determine the combinations of multi-scaling functions for which the desired spline $f(t)$ is produced. This is similar to the corresponding result in the scalar wavelet case [10] where the left eigenvectors of the infinite matrix H give the combinations of scaling functions for which polynomials of degree up to p-1 are produced.

3. ALPERT'S MULTI-WAVELETS

All Alpert's multi-scaling functions of multiplicity r live on the interval [0 1] and they have approximation order p = r. They are all defined by 2 r by r matrix coefficients c_0 and c_1. In this section the construction of Alpert's multi-scaling functions and wavelets are presented for r=2, r=3, r=4. The results are given in Table 1.

3.1 Multiplicity *r* = 2.

This case is characterized by two multi-scaling functions $\phi_0(t)$ and $\phi_1(t)$ which are supported on the interval [0 1]. In order to construct ϕ_0 and ϕ_1 it is required that a linear combination of ϕ_0 and ϕ_1 produce the Hat function. This requirement is written as

$$H(t) = a_0^0\phi_0(t) + a_1^0\phi_0(t-1) + a_0^1\phi_1(t) + a_1^1\phi_1(t-1) \quad (8)$$

Hat function is characterized by the sequence $h[l]$=[½ 1 ½], thus equation (5) can be expressed as

$$\begin{pmatrix} c_0^T & 0 \\ c_1^T & 0 \\ 0 & c_0^T \\ 0 & c_1^T \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I & 0 \\ I & \frac{1}{2}I \\ \frac{1}{2}I & I \\ 0 & \frac{1}{2}I \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \quad (9)$$

where x_0=[$a_0^0\ \ a_0^1$]^T, x_1=[$a_1^0\ \ a_1^1$]^T, I is the 2 by 2 identity matrix.

The orthogonality of multi-scaling functions require

$$HH^T = 2I \quad (10)$$

Solving (9) and (10) numerically provides us with a parameterized set of solutions. Fig.1 shows the scaling functions for multiplicity 2 case. Multi-wavelets are constructed to be orthogonal to the scaling functions and are depicted in Fig.1 as well.

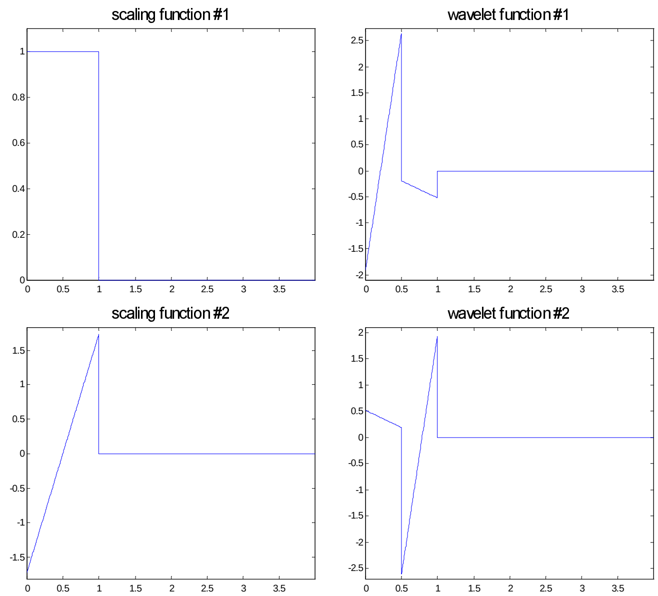

Fig.1. Scaling and Wavelet functions for r=2.

Alpert's multi-scaling functions with multiplicity 2 can interpolate linear functions. In other words a linear combination of multi-scaling functions $\phi_0(t)$ and $\phi_1(t)$ can produce the linear functions because they produce the Hat function and any linear function can be produced by adding shifted versions of the Hat function.

3.2 Multiplicity *r* = 3.

When the multiplicity is 3, the spline with approximation order 3 is defined by the sequence $h[l]$=[¼ ¾ ¾ ¼], thus equation (5) becomes

$$\begin{pmatrix} c_0^T & 0 & 0 \\ c_1^T & 0 & 0 \\ 0 & c_0^T & 0 \\ 0 & c_1^T & 0 \\ 0 & 0 & c_0^T \\ 0 & 0 & c_1^T \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}I & 0 & 0 \\ \frac{3}{4}I & \frac{1}{4}I & 0 \\ \frac{3}{4}I & \frac{3}{4}I & \frac{1}{4}I \\ \frac{1}{4}I & \frac{3}{4}I & \frac{3}{4}I \\ 0 & \frac{1}{4}I & \frac{3}{4}I \\ 0 & 0 & \frac{1}{4}I \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \quad (11)$$

where x_0=[$a_0^0\ a_0^1\ a_0^2$]^T, x_1=[$a_1^0\ a_1^1\ a_1^2$]^T, x_2=[$a_2^0\ a_2^1\ a_2^2$]^T and I is the 3 by 3 identity matrix.

Fig. 2 shows the scaling functions and wavelets. The matrices c_0 and c_1 and vecors x_0, x_1 and x_2 are given in Table 1.

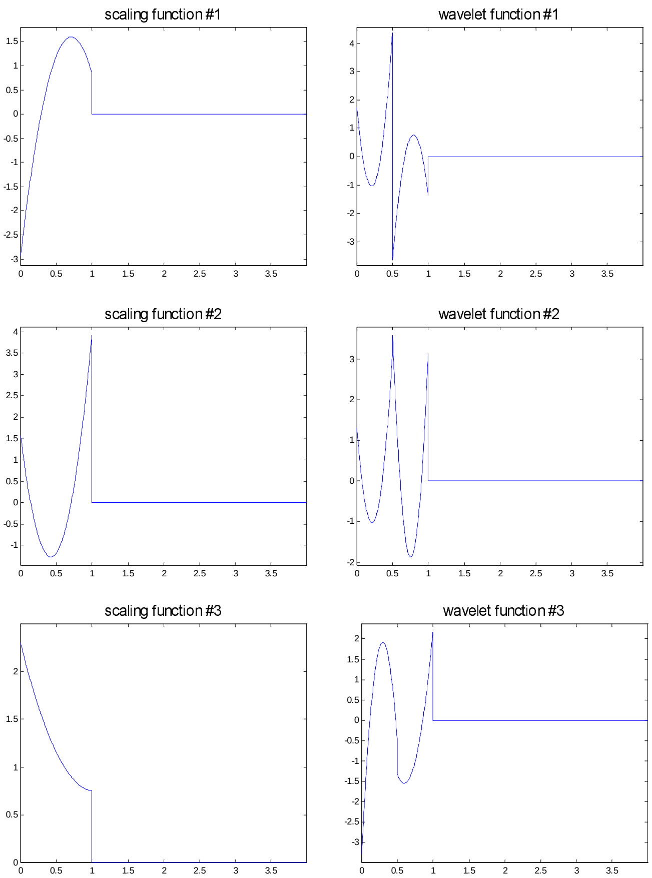

Fig. 2 Scaling and Wavelet functions for r=3.

3.3 Multiplicity r = 4.

Similarly when the multiplicity is 4, equation (5) can be expressed as

$$\begin{pmatrix} c_0^T & 0 & 0 & 0 \\ c_1^T & 0 & 0 & 0 \\ 0 & c_0^T & 0 & 0 \\ 0 & c_1^T & 0 & 0 \\ 0 & 0 & c_0^T & 0 \\ 0 & 0 & c_1^T & 0 \\ 0 & 0 & 0 & c_0^T \\ 0 & 0 & 0 & c_1^T \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} \frac{1}{8}I & 0 & 0 & 0 \\ \frac{4}{8}I & \frac{1}{8}I & 0 & 0 \\ \frac{6}{8}I & \frac{4}{8}I & \frac{1}{8}I & 0 \\ \frac{4}{8}I & \frac{6}{8}I & \frac{4}{8}I & \frac{1}{8}I \\ \frac{1}{8}I & \frac{4}{8}I & \frac{6}{8}I & \frac{4}{8}I \\ 0 & \frac{1}{8}I & \frac{4}{8}I & \frac{6}{8}I \\ 0 & 0 & \frac{1}{8}I & \frac{4}{8}I \\ 0 & 0 & 0 & \frac{1}{8}I \end{bmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (12)$$

where x_0=[a_0^0 a_0^1 a_0^2 a_0^3]^T, x_1=[a_1^0 a_1^1 a_1^2 a_1^3]^T, x_2=[a_2^0 a_2^1 a_2^2 a_2^3]^T, x_2=[a_3^0 a_3^1 a_3^2 a_3^3]^T and I is the 4 by 4 identity matrix. Fig. 3 shows the four scaling functions and wavelets. The matrices c_0 and c_1 and vecors x_0, x_1, x_2 and x_3 are given in Table 1.

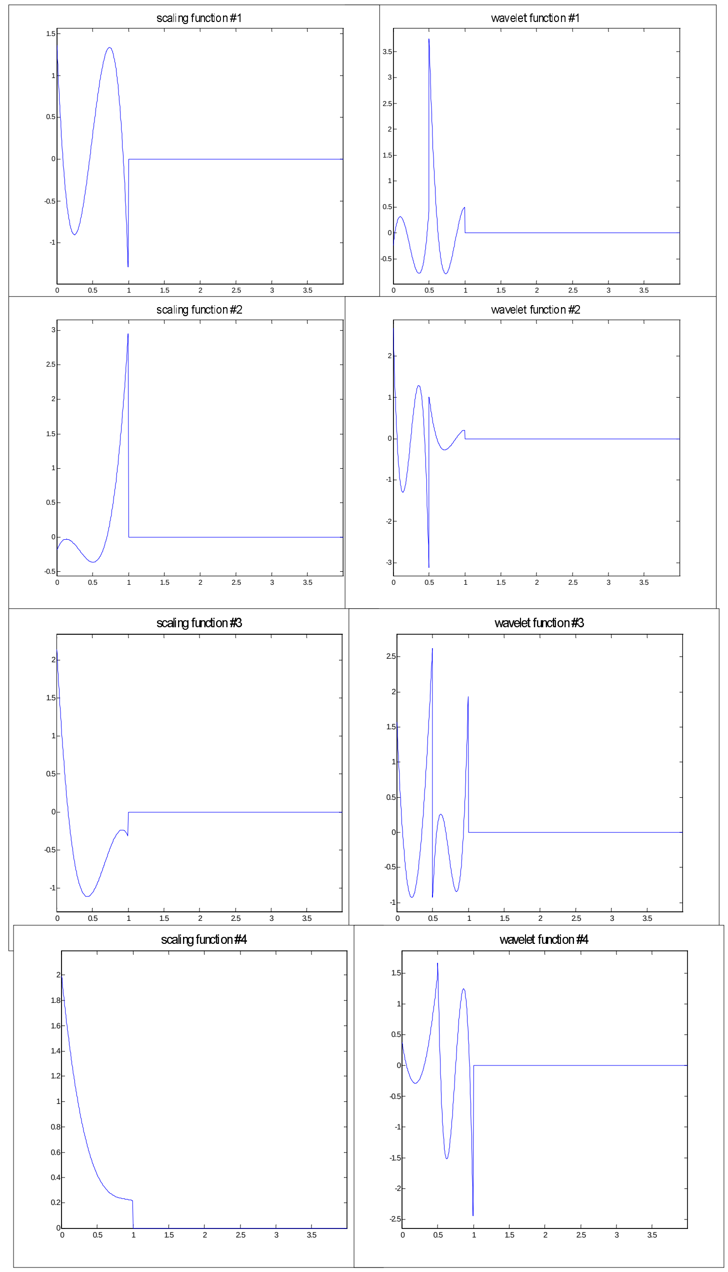

Fig.3. Scaling and Wavelet functions for r=4.

r									
r=2		k=0		k=1					
		$x_0 = (0.1\ 0.057735)^T$							
		$x_1 = (0.1\ -0.057735)^T$							
	c_k	0.7071	0	0.7071	0				
		-0.6124	0.3536	0.6124	0.3536				
	d_k	0.2500	0.9330	-0.2500	-0.0670				
		0.2500	-0.0670	-0.2500	0.9330				
r=3		$x_0 = (0.148857\ 0.116671\ 0.109895)^T$							
		$x_1 = (0.247716\ 9.574953e{-}006\ 0.607612)^T$							
		$x_2 = (-0.078997\ -0.021956\ 0.202796)^T$							
		k=0			k=1				
	c_k	0.4602	0.2047	-0.4611	0.2353	-0.0312	0.6912		
		-0.4892	-0.0960	-0.0680	0.6627	0.5446	0.1075		
		0.1355	0.0121	0.8732	0.0946	0.0275	0.4575		
	d_k	0.0705	0.7521	0.1240	0.4591	-0.2769	-0.3557		
		-0.0182	0.5635	0.0040	-0.4860	0.6663	0.0434		
		0.7249	-0.2560	0.0714	0.2231	0.4256	-0.4160		
r=4		$x_0 = (0.725771\ 1.702077\ -0.732550\ 0.448354)^T$							
		$x_1 = (5.870056\ 6.937935\ -9.288443\ 8.850047)^T$							
		$x_2 = (1.330075\ 0.377855\ -5.989944\ 14.470650)^T$							
		$x_3 = (-0.183919\ -0.198242\ 0.297218\ 2.000000)^T$							
		k=0				k=1			
	c_k	-0.0983	0.0800	0.5415	-0.0147	0.1483	-0.3444	-0.5213	0.5273
		-0.0851	-0.1124	0.0544	-0.0773	0.3857	0.8400	-0.3382	0.0490
		-0.3459	-0.3673	0.6006	0.3245	0.0104	-0.0200	0.1193	-05145
		0.0602	0.0325	-0.2462	0.9358	0.0422	0.0458	-0.0860	0.2182
	d_k	-0.3468	-0.0282	0.1513	-0.0176	0.0528	0.1740	0.7009	0.5756
		0.8370	-0.3642	0.3173	0.0257	0.0335	0.0772	0.2041	0.1289
		0.1548	0.7352	0.3954	0.1022	-0.3949	0.3114	0.0644	-0.1079
		0.1222	0.4136	0.0494	0.0362	0.8170	-0.2001	0.2347	-0.2182

Table 1. Coefficients of 3 Alpert-like multi-wavelets.

4. Conclusion

A new method for constructing Alpert's multi-wavelets is presented. Equations of approximation order are cast in a matrix form that facilitates easy computation of the multi-filter coefficients and recognizes generalized left eigenvectors of finite matrix H_f as vectors that characterize approximation order. In order to test the new method, Alpert's multi-scaling functions with multiplicity 2, 3 and 4 are constructed. The method is also applicable to higher order Alpert multi-wavelets.

5. REFERENCES

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