THE STOCHASTIC CRB FOR ARRAY PROCESSING IN UNKNOWN NOISE FIELDS

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ABSTRACT

The stochastic Cramér-Rao bound (CRB) plays an important role in array processing because several high-resolution direction-of-arrival (DOA) estimation methods are known to achieve this bound asymptotically. In this paper, we study the stochastic CRB on DOA estimation accuracy in the general case of arbitrary unknown noise field parametrized by a vector of unknowns. We derive explicit closed-form expressions for the CRB and examine its properties theoretically and by representative numerical examples.

1. INTRODUCTION

Deterministic and stochastic CRB's play an important role in array processing because the performances of numerous high-resolution DOA estimation methods are known to be comparable to these bounds under certain mild conditions [1]. Moreover, the stochastic CRB can be achieved asymptotically (at a large number of samples) by several methods, such as stochastic maximum likelihood (ML) [2], MODE [1], and WSF [2].

The deterministic CRB on DOA estimation was derived in [3] for the uniform white noise case. Recently, these results were extended to the general case of an arbitrary unknown noise field [4].

The derivation of the stochastic CRB represents a more challenging task (even in the simplest case of uniform white noise). Such a derivation has been found in an indirect form (i.e., using the asymptotic covariance matrix of the ML estimator) some ten years ago by several authors [1], [2]. Although recently this derivation has been extended to a few particular colored noise models [5], closed-form expressions for the general unknown noise model [4] remain an open problem. Several attempts to obtain the stochastic CRB directly has been made, but such a derivation has been found only recently, both for the uniform and nonuniform white noise cases [6], [7].

In this paper, we derive closed-form expressions for the stochastic CRB in the most general case of an arbitrary unknown noise field. Our derivation extends the proof presented in [6]. We analyze the properties of the obtained bound, discuss several practically important special cases, and present a numerical comparison of the stochastic and deterministic bounds for several relevant white and colored noise models.

2. ARRAY SIGNAL MODEL

Let an array of m sensors receive the signals emitted by n narrowband far-field sources with the DOA's $\{\theta_1, \dots, \theta_n\}$. The $m \times 1$ observation vector can be modeled as [1], [2]

$$\boldsymbol{y}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{e}(t) \tag{1}$$

where $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_n]^T$ is the $n \times 1$ DOA vector, $\boldsymbol{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ is the $n \times 1$ vector of random signal waveforms, $\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), \dots, \boldsymbol{a}(\theta_n)]$ is the $m \times n$ direction matrix, $\boldsymbol{e}(t) = [e_1(t), e_2(t) \dots, e_m(t)]^T$ is the $m \times 1$ vector of sensor noise, $\boldsymbol{a}(\theta)$ is the steering vector, and $(\cdot)^T$ denotes the transpose. Assume the noise and signal vectors $\boldsymbol{e}(t)$ and $\boldsymbol{x}(t)$ to be temporally white zero-mean Gaussian processes with the unknown covariance matrices $\boldsymbol{Q} = \mathcal{E}\{\boldsymbol{e}(t)\boldsymbol{e}^*(t)\}$ and $\boldsymbol{P} = \mathcal{E}\{\boldsymbol{x}(t)\boldsymbol{x}^*(t)\}$, respectively, where $(\cdot)^*$ denotes the conjugate transpose. Hence, the random array observations satisfy the stochastic model $\boldsymbol{y}(t) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R})$ (for example, see [3]), where

$$\boldsymbol{R} = \mathcal{E}\{\boldsymbol{y}(t)\boldsymbol{y}^*(t)\} = \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^* + \boldsymbol{Q} \tag{2}$$

is the $m \times m$ array data covariance matrix.

Let us consider the following general model [4]

$$\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{\sigma}) \tag{3}$$

where $\boldsymbol{\sigma} = [\sigma_1, \dots, \sigma_p]^T$ is the vector of unknown coefficients which are used to parameterize the noise covariance matrix. Thus, the $(n^2+n+p) \times 1$ vector of unknown real parameters can be written as $\boldsymbol{\alpha} = [\boldsymbol{\theta}^T, \boldsymbol{\rho}^T, \boldsymbol{\sigma}^T]^T$, where $\boldsymbol{\rho}$ is the $n^2 \times 1$ vector made from the upper triangle of $\boldsymbol{P}$, i.e. from $\{\boldsymbol{P}_{ii}\}$ and $\{\mathrm{Re}\{\boldsymbol{P}_{il}\}, \mathrm{Im}\{\boldsymbol{P}_{il}\};\ l > i\}$.

3. THE STOCHASTIC CRB

Under the previous assumptions, the Fisher information matrix (FIM) for the parameter vector $\boldsymbol{\alpha}$ is given by

$$\mathrm{FIM}_{i,k} = N\,\mathrm{tr}\left(\frac{d\boldsymbol{R}}{d\alpha_i}\boldsymbol{R}^{-1}\frac{d\boldsymbol{R}}{d\alpha_k}\boldsymbol{R}^{-1}\right) \tag{4}$$

for $i, k = 1, \dots, n^2+n+p$, where N is the number of snapshots. Since in most applications both $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are nuisance parameters, we will be interested only in the $n \times n$ $\boldsymbol{\theta}$-block $\mathrm{CRB}(\boldsymbol{\theta})$ of the full $(n^2+n+p) \times (n^2+n+p)$ CRB matrix FIM^{-1}.

We will make relatively frequent use of the following well-known identities

$$\mathrm{tr}(\boldsymbol{X}\boldsymbol{Y}) = \mathrm{vec}\,(\boldsymbol{X}^*)^*\,\mathrm{vec}\,(\boldsymbol{Y}) \tag{5}$$

$$\mathrm{vec}(\boldsymbol{X}\boldsymbol{Y}\boldsymbol{Z}) = \left(\boldsymbol{Z}^T \otimes \boldsymbol{X}\right)\mathrm{vec}(\boldsymbol{Y}) \tag{6}$$

$$(\boldsymbol{W} \otimes \boldsymbol{X})(\boldsymbol{Y} \otimes \boldsymbol{Z}) = (\boldsymbol{W}\boldsymbol{Y}) \otimes (\boldsymbol{X}\boldsymbol{Z}) \tag{7}$$

This work was partly supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada and by the Swedish Foundation for Strategic Research (SSF).

which hold for any conformable matrices $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ and $\boldsymbol{W}$. Here, $\otimes$ denotes the Kronecker product, and $\mathrm{vec}(\cdot)$ is the operator stacking the columns of a matrix on top of each other. Using (4) along with (5) and (6) yields

$$\frac{1}{N}\mathrm{FIM} = \left(\frac{d\boldsymbol{r}}{d\boldsymbol{\alpha}^T}\right)^* \left(\boldsymbol{R}^{-T} \otimes \boldsymbol{R}^{-1}\right) \left(\frac{d\boldsymbol{r}}{d\boldsymbol{\alpha}^T}\right) \tag{8}$$

where $\boldsymbol{r} = \mathrm{vec}(\boldsymbol{R}) = (\boldsymbol{A}^c \otimes \boldsymbol{A})\mathrm{vec}(\boldsymbol{P}) + \mathrm{vec}(\boldsymbol{Q})$, and $(\cdot)^c$ denotes the complex conjugate. Using the partition

$$\left(\boldsymbol{R}^{-T/2} \otimes \boldsymbol{R}^{-1/2}\right) \left[\frac{d\boldsymbol{r}}{d\boldsymbol{\theta}^T} \,\Big|\, \frac{d\boldsymbol{r}}{d\boldsymbol{\rho}^T} \,,\, \frac{d\boldsymbol{r}}{d\boldsymbol{\sigma}^T}\right] \triangleq [\boldsymbol{G} \mid \boldsymbol{\Delta}] \tag{9}$$

we rewrite (8) as

$$\frac{1}{N}\mathrm{FIM} = \begin{bmatrix} \boldsymbol{G}^* \\ \boldsymbol{\Delta}^* \end{bmatrix} [\boldsymbol{G} \,,\, \boldsymbol{\Delta}] \tag{10}$$

Applying a standard result on the inversion of partitioned matrices, we obtain from (10) that

$$\mathrm{CRB}(\boldsymbol{\theta}) = \frac{1}{N} \left(\boldsymbol{G}^* \boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp} \boldsymbol{G}\right)^{-1} \tag{11}$$

where $\boldsymbol{\Pi}_{\boldsymbol{\Delta}} = \boldsymbol{\Delta}(\boldsymbol{\Delta}^*\boldsymbol{\Delta})^{-1}\boldsymbol{\Delta}^*$ and $\boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp} = \boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{\Delta}}$. Note that the existence of $(\boldsymbol{\Delta}^*\boldsymbol{\Delta})^{-1}$ is guaranteed by the existence of FIM^{-1}. Furthermore, let us partition $\boldsymbol{\Delta}$ as

$$\boldsymbol{\Delta} = \left(\boldsymbol{R}^{-T/2} \otimes \boldsymbol{R}^{-1/2}\right) \left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\rho}^T} \,\Big|\, \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\sigma}^T}\right] \triangleq [\boldsymbol{V} \mid \boldsymbol{U}] \tag{12}$$

As the range of $\boldsymbol{\Delta}$ is the same as the range of $\left[\boldsymbol{V}, \boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{U}\right]$, it follows that

$$\boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp} = \boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp} - \boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{U}\left[\boldsymbol{U}^*\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{U}\right]^{-1}\boldsymbol{U}^*\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp} \tag{13}$$

From (11) and (13) we obtain that

$$\mathrm{CRB}(\boldsymbol{\theta}) = \frac{1}{N}\left(\boldsymbol{F} - \boldsymbol{M}\boldsymbol{T}^{-1}\boldsymbol{M}^*\right)^{-1} \tag{14}$$

where

$$\boldsymbol{F} = \boldsymbol{G}^*\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{G}\,, \quad \boldsymbol{M} = \boldsymbol{G}^*\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{U}\,, \quad \boldsymbol{T} = \boldsymbol{U}^*\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{U} \tag{15}$$

To proceed further we need to evaluate the derivatives of $\boldsymbol{r}$ with respect to $\{\alpha_k\}$. First, we consider $d\boldsymbol{r}/d\boldsymbol{\theta}^T$. Let $\boldsymbol{p}_k$ denote the kth column of $\boldsymbol{P}$, i.e. $\boldsymbol{P} = [\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n]$. Hence,

$$\frac{d\boldsymbol{R}}{d\theta_k} = \boldsymbol{d}_k\boldsymbol{p}_k^*\boldsymbol{A}^* + \boldsymbol{A}\boldsymbol{p}_k\boldsymbol{d}_k^* \tag{16}$$

where $\boldsymbol{d}_k = d\boldsymbol{a}(\theta_k)/d\theta_k$. Therefore, the kth column of $\boldsymbol{G}$ is given by

$$\boldsymbol{g}_k = \mathrm{vec}\left(\boldsymbol{R}^{-1/2}\frac{d\boldsymbol{R}}{d\theta_k}\boldsymbol{R}^{-1/2}\right) \triangleq \mathrm{vec}(\boldsymbol{Z}_k + \boldsymbol{Z}_k^*) \tag{17}$$

$$\boldsymbol{Z}_k = \boldsymbol{R}^{-1/2}\boldsymbol{A}\boldsymbol{p}_k\boldsymbol{d}_k^*\boldsymbol{R}^{-1/2} \tag{18}$$

Next consider $d\boldsymbol{r}/d\boldsymbol{\rho}^T$. The key observation here is that $\mathrm{vec}(\boldsymbol{P}) = \boldsymbol{J}\boldsymbol{\rho}$, where $\boldsymbol{J}$ is a constant nonsingular matrix. To check this, we note that $\mathrm{vec}(\boldsymbol{P})$ is a permuted version of the vector $[\boldsymbol{P}_{11}, \ldots, \boldsymbol{P}_{nn}, \boldsymbol{P}_{12}, \boldsymbol{P}_{12}^*, \ldots, \boldsymbol{P}_{n-1,n}, \boldsymbol{P}_{n-1,n}^*]^T$. It can be readily verified that $\boldsymbol{J}$ is a correspondingly permuted version of the block-diagonal matrix whose first n diagonal blocks are equal to 1 and the remaining diagonal blocks are given by $\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$ (see [6], [7]). Using this observation along with (7) and (12) yields

$$\boldsymbol{V} = \left(\left(\boldsymbol{R}^{-T/2}\boldsymbol{A}^c\right) \otimes \left(\boldsymbol{R}^{-1/2}\boldsymbol{A}\right)\right)\boldsymbol{J} \tag{19}$$

From (11) and (13), we notice that the $\mathrm{CRB}(\boldsymbol{\theta})$ depends on $\boldsymbol{V}$ only via $\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}$. Since $\boldsymbol{J}$ is nonsingular, we obtain from (19) that

$$\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp} = \boldsymbol{\Pi}_{(\boldsymbol{R}^{-1/2}\boldsymbol{A})^c \otimes (\boldsymbol{R}^{-1/2}\boldsymbol{A})}^{\perp} \tag{20}$$

Hence, the explicit form of $\boldsymbol{J}$ in (19) is immaterial. This is an important observation which will simplify the derivation of the $\mathrm{CRB}(\boldsymbol{\theta})$.

Using the well-known identity $(\boldsymbol{W} \otimes \boldsymbol{Z})^{-1} = \boldsymbol{W}^{-1} \otimes \boldsymbol{Z}^{-1}$ (which holds true for any square nonsingular matrices $\boldsymbol{W}$ and $\boldsymbol{Z}$), we obtain from (7) that for any matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ with full column-rank

$$\boldsymbol{\Pi}_{(\boldsymbol{X}\otimes\boldsymbol{Y})}^{\perp} = \boldsymbol{I} \otimes \boldsymbol{\Pi}_{\boldsymbol{Y}}^{\perp} + \boldsymbol{\Pi}_{\boldsymbol{X}}^{\perp} \otimes \boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{X}}^{\perp} \otimes \boldsymbol{\Pi}_{\boldsymbol{Y}}^{\perp} \tag{21}$$

From (17), (20) and (21), we get

$$\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{g}_k = \mathrm{vec}\left(\boldsymbol{\Pi}_{\boldsymbol{R}^{-1/2}\boldsymbol{A}}^{\perp}\boldsymbol{Z}_k^* + \boldsymbol{Z}_k\boldsymbol{\Pi}_{\boldsymbol{R}^{-1/2}\boldsymbol{A}}^{\perp}\right) \tag{22}$$

where the obvious property $\boldsymbol{\Pi}_{\boldsymbol{R}^{-1/2}\boldsymbol{A}}^{\perp}\boldsymbol{Z}_k = \mathbf{O}$ is exploited. Furthermore, we can show that

$$\boldsymbol{R}^{-1/2}\boldsymbol{\Pi}_{\boldsymbol{R}^{-1/2}\boldsymbol{A}}^{\perp}\boldsymbol{R}^{-1/2} = \boldsymbol{Q}^{-1/2}\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-1/2} \tag{23}$$

where $\tilde{\boldsymbol{A}} = \boldsymbol{Q}^{-1/2}\boldsymbol{A}$.

Finally, introduce $\boldsymbol{Q}_k^{'} = d\boldsymbol{R}/d\sigma_k = d\boldsymbol{Q}/d\sigma_k$, $k = 1, \ldots, p$. We have now all ingredients to obtain $\mathrm{CRB}(\boldsymbol{\theta})$. Now, let us derive explicit expressions for the matrices $\boldsymbol{F}$, $\boldsymbol{M}$, and $\boldsymbol{T}$. First, we consider $\boldsymbol{F}$. Using (15) along with (5), (17), and (22) gives

$$\boldsymbol{F}_{i,k} = 2\mathrm{Re}\left\{\mathrm{tr}\left(\boldsymbol{Z}_i\boldsymbol{\Pi}_{\boldsymbol{R}^{-1/2}\boldsymbol{A}}^{\perp}\boldsymbol{Z}_k^*\right)\right\} \tag{24}$$

Inserting (18) into (24) and then using (23) yields

$$\boldsymbol{F}_{i,k} = 2\mathrm{Re}\left\{\left(\boldsymbol{d}_i^*\boldsymbol{Q}^{-1/2}\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-1/2}\boldsymbol{d}_k\right)\left(\boldsymbol{p}_k^*\boldsymbol{A}^*\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{p}_i\right)\right\} \tag{25}$$

Next, we consider the matrix $\boldsymbol{M}$. Making use of (15) along with (5), (6), (12), (18), (22), and (23), we have

$$\boldsymbol{M}_{i,k} = 2\mathrm{Re}\left\{\mathrm{tr}\left(\boldsymbol{Q}_k^{'}\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{p}_i\boldsymbol{d}_i^*\boldsymbol{Q}^{-1/2}\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-1/2}\right)\right\} \tag{26}$$

At last, we consider the matrix $\boldsymbol{T}$. Using (15) along with (5), (6), (12), (20), (21), and (23), it readily follows that

$$\begin{aligned}\boldsymbol{T}_{i,k} &= 2\,\mathrm{Re}\left\{\mathrm{tr}\left(\boldsymbol{Q}_i^{'}\boldsymbol{Q}^{-1/2}\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-1/2}\boldsymbol{Q}_k^{'}\boldsymbol{R}^{-1}\right)\right\} \\ &= \mathrm{tr}\left(\boldsymbol{Q}_i^{'}\boldsymbol{Q}^{-1/2}\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-1/2}\boldsymbol{Q}_k^{'}\boldsymbol{Q}^{-1/2}\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-1/2}\right)\end{aligned} \tag{27}$$

From (25)-(27) we obtain the following closed-form expressions for the DOA-related block of CRB:

$$\begin{aligned}\mathrm{CRB}(\boldsymbol{\theta}) &= \frac{1}{N}\left(\boldsymbol{F} - \boldsymbol{M}\boldsymbol{T}^{-1}\boldsymbol{M}^T\right)^{-1} \\ \boldsymbol{F} &= 2\mathrm{Re}\left\{\left(\tilde{\boldsymbol{D}}^*\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}^{\perp}\tilde{\boldsymbol{D}}\right) \odot \left(\boldsymbol{P}\tilde{\boldsymbol{A}}^*\tilde{\boldsymbol{R}}^{-1}\tilde{\boldsymbol{A}}\boldsymbol{P}\right)^T\right\} \\ \boldsymbol{M}_{i,k} &= 2\mathrm{Re}\left\{\tilde{\boldsymbol{d}}_i^*\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}^{\perp}\tilde{\boldsymbol{Q}}_k^{'}\tilde{\boldsymbol{R}}^{-1}\tilde{\boldsymbol{A}}\boldsymbol{p}_i\right\} \\ \boldsymbol{T}_{i,k} &= 2\mathrm{Re}\left\{\mathrm{tr}\left(\tilde{\boldsymbol{Q}}_i^{'}\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}^{\perp}\tilde{\boldsymbol{Q}}_k^{'}\tilde{\boldsymbol{R}}^{-1}\right)\right\} - \mathrm{tr}\left(\tilde{\boldsymbol{Q}}_i^{'}\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}^{\perp}\tilde{\boldsymbol{Q}}_k^{'}\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}^{\perp}\right)\end{aligned} \tag{28}$$

where $\boldsymbol{D} = [\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_n]$, $\tilde{\boldsymbol{R}} = \boldsymbol{Q}^{-1/2}\boldsymbol{R}\boldsymbol{Q}^{-1/2}$, $\tilde{\boldsymbol{D}} = \boldsymbol{Q}^{-1/2}\boldsymbol{D}$, $\tilde{\boldsymbol{Q}}'_k = \boldsymbol{Q}^{-1/2}\boldsymbol{Q}'_k\boldsymbol{Q}^{-1/2}$, and $\odot$ denotes the Schur-Hadamard product. It can be readily checked that the last term in the fourth equation of (28) is real-valued.

The expressions (28) can be reformulated in a more explicit matrix form. Let us introduce the following auxiliary $n^2 \times n$ and $m^2 \times p$ matrices $\boldsymbol{\mathcal{Q}} = [\mathrm{vec}(\boldsymbol{e}_1\boldsymbol{e}_1^T), \mathrm{vec}(\boldsymbol{e}_2\boldsymbol{e}_2^T), \ldots, \mathrm{vec}(\boldsymbol{e}_n\boldsymbol{e}_n^T)]$ and $\boldsymbol{\mathcal{P}} = [\mathrm{vec}(\tilde{\boldsymbol{Q}}'_1), \mathrm{vec}(\tilde{\boldsymbol{Q}}'_2), \ldots, \mathrm{vec}(\tilde{\boldsymbol{Q}}'_p)]$, respectively, where the vector $\boldsymbol{e}_i$ contains one in the ith position and zeros elsewhere. Using these matrices and the properties (5) and (6), we can rewrite $\boldsymbol{M}$ and $\boldsymbol{T}$ as

$$\begin{aligned} \boldsymbol{M} &= 2\mathrm{Re}\left\{\boldsymbol{\mathcal{Q}}^T\left(\tilde{\boldsymbol{D}}^*\boldsymbol{\Pi}^{\perp}_{\tilde{\boldsymbol{A}}}\right)\otimes\left(\boldsymbol{P}^T\tilde{\boldsymbol{A}}^T\tilde{\boldsymbol{R}}^{-T}\right)\boldsymbol{\mathcal{P}}^c\right\} \\ \boldsymbol{T} &= 2\mathrm{Re}\left\{\boldsymbol{\mathcal{P}}^*\left(\tilde{\boldsymbol{R}}^{-T}\otimes\boldsymbol{\Pi}^{\perp}_{\tilde{\boldsymbol{A}}}\right)\boldsymbol{\mathcal{P}}\right\} - \boldsymbol{\mathcal{P}}^*\left((\boldsymbol{\Pi}^{\perp}_{\tilde{\boldsymbol{A}}})^T\otimes\boldsymbol{\Pi}^{\perp}_{\tilde{\boldsymbol{A}}}\right)\boldsymbol{\mathcal{P}} \end{aligned}$$

4. DISCUSSION

Equation (11) gives a nice geometrical interpretation of the stochastic CRB. This interpretation is similar to that of the deterministic bound [8]. From (11) it follows that any extra parameters added to $\boldsymbol{\sigma}$ increase the dimension of the subspace $<\boldsymbol{\Delta}>$ and, therefore, increase the stochastic CRB as well. This means that extra nuisance parameters can only reduce the potential DOA estimation performance.

Equation (10) determines necessary conditions for existence of the CRB. As a matter of fact, from (10) it follows that the FIM is nonsingular if the $m^2 \times (n^2 + n + p)$ matrix $[\boldsymbol{G}, \boldsymbol{\Delta}]$ is full column-rank. Hence, the necessary condition is

$$p \leq m^2 - n^2 - n \tag{29}$$

For example, consider the case where there is no *a priori* information about the noise covariance matrix. Using its Hermitian structure, however, it can be parametrized by means of $p = m^2$ real parameters. Apparently, in this case (29) is not satisfied and, therefore, such a straightforward parameterization of the noise covariance matrix leads to a singular FIM.

Let us compare the derived expressions to the well-known results for the stochastic CRB in the uniform noise case. In this case, $\boldsymbol{Q} = \sigma\boldsymbol{I}$, where σ is the noise variance. The uniform bound is given by [1], [2]

$$\mathrm{CRB}_{\mathrm{U}}(\boldsymbol{\theta}) = \frac{\sigma}{2N}\left(\mathrm{Re}\left\{(\boldsymbol{D}^*\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}}\boldsymbol{D})\odot\left(\boldsymbol{P}\boldsymbol{A}^*\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{P}\right)^T\right\}\right)^{-1} \tag{30}$$

Consider the situation when the noise is uniform and spatially white, but modeled using the general model (3) with $p > 1$ noise parameters. Comparing (30) with (28) under these conditions, we have $\frac{1}{N}\boldsymbol{F}^{-1}|_{\boldsymbol{Q}=\sigma\boldsymbol{I}} = \mathrm{CRB}_{\mathrm{U}}(\boldsymbol{\theta})$. Furthermore, according to (15), $\boldsymbol{T}^{-1}$ is nonnegative definite, and hence $\boldsymbol{F} - \boldsymbol{M}\boldsymbol{T}^{-1}\boldsymbol{M}^T \leq \boldsymbol{F}$. Therefore, we obtain that

$$\mathrm{CRB}(\boldsymbol{\theta})\Big|_{\boldsymbol{Q}=\sigma\boldsymbol{I}} \geq \mathrm{CRB}_{\mathrm{U}}(\boldsymbol{\theta}) \tag{31}$$

We stress that this result is quite different from that in the deterministic case [1], [3] where $\mathrm{CRB}(\boldsymbol{\theta})|_{\boldsymbol{Q}=\sigma\boldsymbol{I}} = \mathrm{CRB}_{\mathrm{U}}(\boldsymbol{\theta})$ [7]. To explain this difference, note that in the stochastic case the noise and signal parameters are not decoupled as in the deterministic case. In fact, the inequality (31) verifies the so-called *parsimony principle*.

Let us now compare the stochastic and asymptotic deterministic CRB's in the general case. The latter bound is given by $\mathrm{CRB}_{\mathrm{DET}}(\boldsymbol{\theta}) = \frac{1}{2N}(\mathrm{Re}\{(\tilde{\boldsymbol{D}}^*\boldsymbol{\Pi}^{\perp}_{\tilde{\boldsymbol{A}}}\tilde{\boldsymbol{D}})\odot\boldsymbol{P}^T\})^{-1}$. The following inequality can be proven

$$\mathrm{CRB}(\boldsymbol{\theta}) \geq \frac{1}{N}\boldsymbol{F}^{-1} \geq \mathrm{CRB}_{\mathrm{DET}}(\boldsymbol{\theta}) \tag{32}$$

where the second part of (32) is derived in [1], and the first part follows from (28) together with the fact that the matrix $\boldsymbol{M}\boldsymbol{T}^{-1}\boldsymbol{M}^T$ is nonnegative definite.

From (32) we may expect that in the colored and nonuniform noise cases there may be a more essential difference between the deterministic and stochastic CRB's compared to the uniform white noise case, because the inequality $\mathrm{CRB}(\boldsymbol{\theta}) \geq \mathrm{CRB}_{\mathrm{DET}}(\boldsymbol{\theta})$ is "strengthened" by the additional term $-\boldsymbol{M}\boldsymbol{T}^{-1}\boldsymbol{M}^T$. This conjecture will be verified in the next section by several numerical examples.

4.1. The Nonuniform White Noise Case

In this case $\boldsymbol{Q}$ is modeled as a diagonal matrix $\boldsymbol{Q}(\boldsymbol{\sigma}) = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, which implies that $p = m$ and $\boldsymbol{Q}'_i = \boldsymbol{e}_i\boldsymbol{e}_i^T$. It can be proven that (28) can be transformed to

$$\mathrm{CRB}(\boldsymbol{\theta}) = \frac{1}{N}\left(\boldsymbol{F} - \boldsymbol{H}\boldsymbol{K}^{-1}\boldsymbol{H}^T\right)^{-1} \tag{33}$$

with $\boldsymbol{H} = 2\mathrm{Re}\{(\tilde{\boldsymbol{D}}^*\boldsymbol{\Pi}^{\perp}_{\tilde{\boldsymbol{A}}})\odot(\tilde{\boldsymbol{R}}^{-1}\tilde{\boldsymbol{A}}\boldsymbol{P})^T\}$ and $\boldsymbol{K} = 2\mathrm{Re}\{\boldsymbol{\Pi}^{\perp}_{\tilde{\boldsymbol{A}}}\odot\tilde{\boldsymbol{R}}^{-T}\} - \boldsymbol{\Pi}^{\perp}_{\tilde{\boldsymbol{A}}}\odot(\boldsymbol{\Pi}^{\perp}_{\tilde{\boldsymbol{A}}})^T$. Note that these expressions are identical to the results derived in [7], yet the matrix $\boldsymbol{K}$ in [7] is written in the equivalent form $\boldsymbol{K} = \tilde{\boldsymbol{R}}^{-T}\odot\tilde{\boldsymbol{R}}^{-1} - (\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}\tilde{\boldsymbol{R}}^{-1})^T\odot(\boldsymbol{\Pi}_{\tilde{\boldsymbol{A}}}\tilde{\boldsymbol{R}}^{-1})$.

4.2. The Uniform White Noise Case

In this case, we have only one noise nuisance parameter ($p = 1$), and the vector $\boldsymbol{\sigma}$ becomes a scalar, σ. The model (3) can be simplified as $\boldsymbol{Q}(\sigma) = \sigma\boldsymbol{I}$ and $\boldsymbol{M}$ becomes the $n \times 1$ vector $\boldsymbol{m}$. We easily obtain that $\boldsymbol{m} = \boldsymbol{0}$ and (28) simplifies to (30).

5. NUMERICAL EXAMPLES

We assume a uniform linear array (ULA) of $m = 10$ sensors spaced a half wavelength apart and two equi-powered uncorrelated sources with the DOA's $\theta_1 = 7^\circ$ and $\theta_2 = 13^\circ$. $N = 100$ is taken.

Our first two examples correspond to the following colored noise field models [9], [10]:

$$[\boldsymbol{Q}]_{i,k} = \sigma\exp\{-(i-k)^2\zeta\} \tag{34}$$

$$[\boldsymbol{Q}]_{i,k} = \sigma\exp\{-|i-k|\zeta\} \tag{35}$$

respectively, where ζ is the noise "color" parameter and $\boldsymbol{\sigma} = [\sigma, \zeta]^T$. In both examples, $\mathrm{SNR} = 10\log_{10}(\sigma_S/\sigma) = 0$ dB. Fig. 1 shows the deterministic and stochastic CRB's versus ζ for the first and second examples. The deterministic CRB is averaged over 100 simulation runs.

In the third example, we model the sensor noise as a white nonuniform process with the diagonal covariance matrix [7]

$$\boldsymbol{Q} = \mathrm{diag}\{\sigma+\delta_1, \sigma+\delta_2, \ldots, \sigma+\delta_m\} \tag{36}$$

where $\{\delta_i\}_{i=1}^m$ are independently drawn from the uniform random generator in the interval $[0, \beta]$ and the "initial" SNR (which is

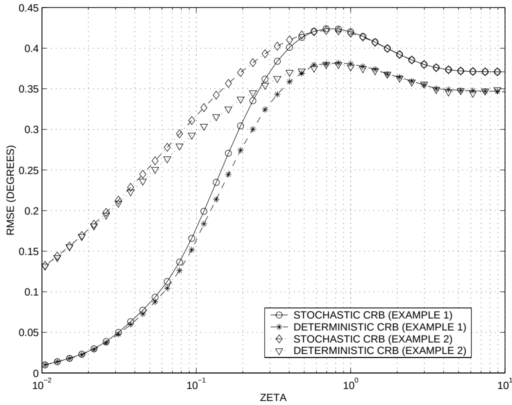

Fig. 1. The CRB's versus ζ. First and second examples.

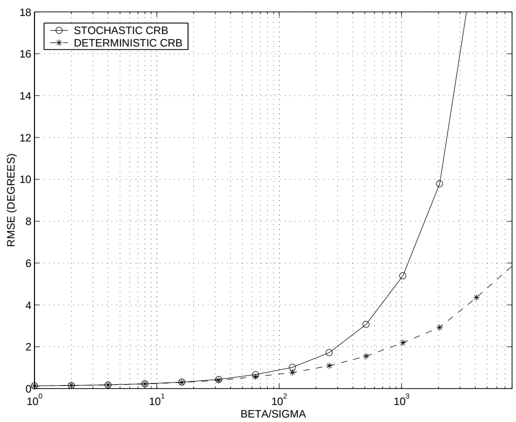

Fig. 2. The CRB's versus β/σ. Third example.

equal to $10\log_{10}(\sigma_S/\sigma)$) is $= -10$ dB. Fig. 2 displays the deterministic and stochastic CRB's versus the parameter β/σ. Both curves are averaged over 100 simulation runs.

In the fourth example, we model the noise as an AR process [4], so that $\boldsymbol{Q} = \sigma(\boldsymbol{C}_1\boldsymbol{C}_1^* - \boldsymbol{C}_2\boldsymbol{C}_2^*)^{-1}$, where the triangular Toeplitz matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are parametrized by the AR coefficients $\{a_1, \ldots, a_{M-1}\}$, M is the order of the AR model, and $\boldsymbol{\sigma} = [\sigma, a_1, \ldots, a_{M-1}]^T$ (for more details, see [4]). We assume that $\mathrm{SNR} = -5$ dB and vary M so that the AR coefficients corresponding to the ith model order are given by the $i \times 1$ subvector composed of the first i elements of the vector $[1, -0.88, 0.84, -0.86, 0.85, -0.88, 0.84, -0.85, 0.83, -0.82]^T$. Fig. 3 shows the stochastic and averaged deterministic CRB's versus M.

From Figs. 1-3, we see that both in the colored and nonuniform white noise cases, the stochastic bound can visibly exceed the deterministic one. In certain cases (for example, see Fig. 2), the difference between the stochastic and deterministic bounds may be dramatic. This, along with the well-known fact that the deterministic bound is too optimistic and nonachievable, verifies the importance of the derived expressions for array processing applications.

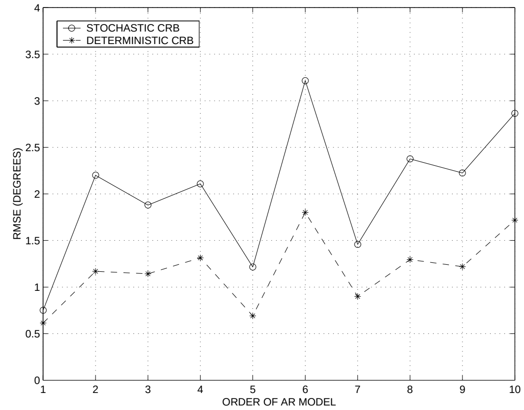

Fig. 3. The CRB's versus the order of AR model. Fourth example.

6. ACKNOWLEDGEMENT

The authors are grateful to Prof. L.L. Scharf who suggested the geometrical interpretation of (11).

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