APPROXIMATE CFAR SIGNAL DETECTION IN STRONG LOW RANK NON-GAUSSIAN INTERFERENCE

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ABSTRACT

Recent work suggests that the performance of conventional Gaussian-based adaptive methods can degrade severely in correlated non-Gaussian interference. We have addressed this problem by developing a new generalized likelihood ratio test (GLRT) for detecting a signal in unknown, strong non-Gaussian low rank interference plus white Gaussian noise which does not need detailed knowledge of the non-Gaussian distribution. The optimality of the proposed GLRT detector is established using perturbation expansions of the test statistic to show that it is closely related to the UMPI test for this problem. Computer simulations indicate that the new detector significantly outperforms standard adaptive methods in non-Gaussian interference and is robust.

1. INTRODUCTION

Non-Gaussian disturbances have been observed in diverse applications such as radar, sonar, digital communications, and radio astronomy. Signal detection in unknown colored noise backgrounds has traditionally been accomplished using adaptive methods based on the Gaussian model, whether or not the noise is actually Gaussian distributed. However, recent work has shown that the performance of adaptive detectors based on the Gaussian model can degrade severely when operating in correlated non-Gaussian noise backgrounds [1]. As an example, we computer simulated the invariant matched subspace detector (MSD) of Scharf et. al. [2] in noise consisting of a strong, highly correlated rank-2 compound-Gaussian component embedded in white Gaussian noise noise. Two versions were considered: the optimum MSD that knows the true interference subspace and, motivated by the Principal Component Inverse (PCI) method [3], an adaptive MSD (ASD) that uses an estimate of the interference subspace obtained from signal-free training data. As a reference, we also evaluated the ASD using pure Gaussian noise that had the same nominal covariance matrix as in the non-Gaussian case. The results for all three cases are plotted in figure 1. As clearly seen, the performance of the ASD degrades substantially in the non-Gaussian noise, whereas, the adaptive detector in pure Gaussian noise has performance close to that of the optimum MSD. The effect of non-Gaussian interference on the PCI and subspace methods is further discussed in [4].

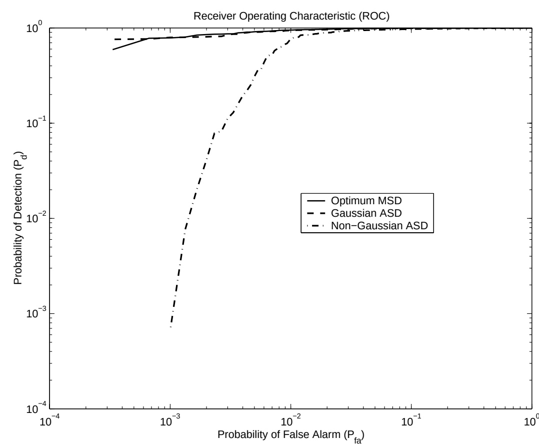

Figure 1: Experimentally measured ROC curves comparing the performance of the detectors at a signal-to-interference ratio of -5 dB.

The underlying problem of designing detectors for non-Gaussian clutter is that in most applications there exists no single family of multivariate non-Gaussian pdfs that accurately characterizes the clutter in all scenarios and environments. We now propose an alternative approach. Instead of trying to model the detailed non-Gaussian statistical characteristics of the noise components, we treat their waveforms as unknown (unknown parameters), but deterministic. Then the detection problem can be re-formulated as a composite hypothesis testing problem [5]. The advantage is that this detection problem is often easier to solve than the original non-Gaussian problem.

More precisely, we model the received complex-valued $m \times 1$ noise plus signal space-time data snapshot at time t_k as a superposition

$$\mathbf{z}_k = \underbrace{\sum_{j=1}^{r_n} a_j^k \mathbf{b}_j}_{subspace\ interference} + \underbrace{c^k}_{signal} \mathbf{s} + \underbrace{\mathbf{n}_k}_{background\ white\ noise} \tag{1}$$

of a strong subspace non-Gaussian interference component and a background white Gaussian noise component $\mathbf{n}_k$, and possibly a signal component. The a_j^k and c^k are the noise and signal expansion coefficients respectively and the $\mathbf{b}_j$ and $\mathbf{s}$ are the noise and signal basis vectors respectively. The non-Gaussianity of the noise is modeled as arising from the expansion coefficients a_j^k rather than the basis vectors $\mathbf{b}_j$. For convenience, a rank-1 signal is assumed.

For the case of known $\mathbf{b}_j$, but unknown a_j^k with unknown multivariate pdf and unknown white noise variance, it is reasonable to seek a test which is invariant to these parameters. Ideally, we desire a uniformly most powerful invariant (UMPI) test [5] (the UMPI test maximizes the probability of detection regardless of the parameter values while keeping the false alarm rate less than or equal to some specified value). Scharf et. al. [2] showed that for data of the form (1) with known interference and signal subspaces, the UMPI test, referred to as the matched subspace detector (MSD), is (in simplified form)

$$\frac{\| P_{P_B^\perp S}\, \mathbf{z} \|_F^2}{\| P_{BS}^\perp\, \mathbf{z} \|_F^2} \begin{array}{c} \mathcal{H}_1 \\ > \\ \leq \\ \mathcal{H}_0 \end{array} \lambda \qquad (2)$$

where λ is some threshold. The matrix $P_{P_B^\perp S}$ is the projection operator onto the part of the signal that remains after the subspace interference has been nulled and $P_{BS}^\perp$ is the projection operator that nulls out both the subspace interference and signal component. Mathematically, $P_{P_B^\perp S}$ and $P_{BS}^\perp$ are given by

$$P_{P_B^\perp S} = P_B^\perp S (S^H P_B^\perp S)^{-1} S^H P_B^\perp \qquad (3)$$

and

$$P_{BS}^\perp = I - [B|S]([B|S]^H [B|S])^{-1} [B|S]^H \qquad (4)$$

where $B = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{r_n}]$ and $S = s$. The matrix $[B|S]$ is obtained by concatenating B and S columnwise) respectively, and

$$P_B^\perp = I - B(B^H B)^{-1} B^H \qquad (5)$$

Test (2) is maximally invariant to scalings of the data and rotations in the column space of B. Hence it is CFAR with respect to the background noise level. It is emphasized since (2) is UMPI, no other CFAR test can perform better.

Although test (2) is optimum, it is difficult to realize because the interference subspace B is seldom known beforehand in practice. One approach is to use the methodology of the PCI method [3] and estimate the unknown interference subspace from a set of signal-free training data. However, as the previous and upcoming numerical examples indicate, this approach may not be optimum when the low rank noise is non-Gaussian.

The approach we take is to treat B, a_j^k and the white noise variance as unknown, but deterministic, and derive the GLRT (the GLRT is obtained by replacing the unknown parameters in the likelihood-ratio test by their ML estimates). Our motivation is that in certain instances, the GLRT can actually be UMPI and often leads to a reasonable or good test [2].

2. NEW GLRT DETECTOR

A secondary data set of K signal-free data vectors is assumed available for training, stacked column-wise into a $m \times K$ matrix X. Detection of the signal is to be performed on a primary data set, consisting of a single data snapshot, denoted as Y. Under the null hypotheses $\mathcal{H}_0$ and signal present hypotheses $\mathcal{H}_1$, the observed data matrices $Z = [X|Y]$ are modeled as

$$\begin{aligned} \mathcal{H}_0 : Z &= BA + N \quad \text{(noise only)} & (6) \\ \mathcal{H}_1 : Z &= BA + [0|Sc] + N \quad \text{(signal + noise)} & (7) \end{aligned}$$

where B is a $m \times r_n$ matrix whose columns generate the low rank interference space, A is a $r_n \times K+1$ matrix whose elements contain the low rank interference expansion coefficients, S is a n x 1 signal replica, and c is the signal amplitude. The elements of matrix N are modeled as IID complex Gaussian random variables with zero-mean and variance σ^2. S is assumed known, but A, B, c, and σ^2 are assumed to be unknown, but deterministic.

A GLRT statistic for the hypothesis testing problem of (6) and (7) is then

$$y \equiv \frac{\overset{max}{B_1, A_1, c, \sigma_1^2} \; (\sigma_1^2)^{-mK} \; e^{-\frac{1}{\sigma_1^2}\|Z - B_1 A_1 - [0|Sc]\|_F^2}}{\overset{max}{B_0, A_0, \sigma_0^2} \; (\sigma_0^2)^{-mK} \; e^{-\frac{1}{\sigma_0^2}\|Z - B_0 A_0\|_F^2}} \qquad (8)$$

which simplifies to the ratio of fitting errors

$$y \equiv \frac{\min B_0, A \; \|Z - B_0 Z_0\|_F^2}{\min B_1, B_1, c \; \|Z - B_1 A_1 - [0|Sc]\|_F^2} \qquad (9)$$

The numerator of (9) is the square-error in fitting the matrix Z by a rank r_n matrix and can be easily evaluated using the SVD of Z. Similarly, the denominator of (9) is the error in jointly fitting Z by a rank r_n matrix and the linear part $[0|Sc]$. However, it can not be directly evaluated using the SVD of Z.

To numerically evaluate the denominator, we propose a criss-cross regression-like method. The idea is to linearize the minimization by holding, say B, constant and then minimizing with respect to only A and c. This is a standard linear least-squares fitting problem and is easy to solve. The procedure is then repeated, this time replacing A with its estimate from the previous step and now minimizing with respect to B and the c. These steps are repeated until convergence.

3. RELATIONSHIP TO UMPI DETECTOR

We now establish the connection of the proposed GLRT to the UMPI matched subspace detector of Scharf et al. [2] by deriving a simple approximation to the test statistic. First, in order to make the comparison, we need to extend the single data vector optimum MSD (2) to the multiple data vector case of (6) and (7). This is simple to do and by substitution (by concatenating all the columns of Z into one vector), we obtain the optimum MSD test statistic for the multiple data vector case:

$$y_{MSD} - 1 \equiv \frac{\| P_{S'} \, \mathbf{z}' \|_F^2}{\| P_{S'}^{\perp} \, \mathbf{z}' \|_F^2} \tag{10}$$

where $\mathbf{z}' = vec(P_B^{\perp} Z)$, $S' = [\, vec(P_B^{\perp}[0|S]),$
$P_{S'} = S'(S'^H S')^{-1} S'$, and $P_{S'}^{\perp} = I - P_{S'}$.

We now use a first-order perturbation expansion for the SVD of a data matrix [6] to obtain an approximation to the GLRT test statistic (9) which can be related to the UMPI MSD (10). In the analysis, both Sc and N are regarded as perturbations and weak relative to BA. The specific derivation details are shown in appendix A. The final approximation for the GLRT statistic derived in appendix A is

$$y - 1 \approx \frac{\| P_{S''} \, \mathbf{z}'' \|_F^2}{\| P_{S''}^{\perp} \, \mathbf{z}'' \|_F^2} \tag{11}$$

where $\mathbf{z}'' = vec(P_B^{\perp} Z P_A^{\perp})$, $S'' = vec(P_B^{\perp}[0|S]P_A^{\perp})$, $P_{S''} = S''(S''^H S'')^{-1} S''$, $P_{S''}^{\perp} = I - P_{S''}$, and
$P_A^{\perp} = I - A^H (AA^H)^{-1} A$.

The only difference between the UMPI MSD (10) and the new GLRT (11) is the post multiplication of the data matrix Z by $P_A^{\perp}$. Thus to first-order, the new GLRT is approximately equivalent to the optimum MSD. By inspection, it is seen that (11) is invariant with respect to common scalings of the columns of the data matrix Z, and thus the background noise level. Thus, the new GLRT is at least approximately CFAR with respect to the background noise level.

When the interference is strong and signal weak, the loss in performance of the GLRT comes from the additional nulling due to the post-multiplication of the data matrix Z by $P_A^{\perp}$. This loss can be interpreted as arising from having to estimate the interference subspace and is a function of the orthogonality of the interference matrix row space to the row space of the signal matrix $[0|Sc]$

4. NUMERICAL EXAMPLES

We now present some numerical examples where a 20 element array is used to detect a weak monochromatic signal embedded in strong, highly correlated, but heavy tailed, rank-2 compound-Gaussian clutter plus white complex Gaussian noise. The rank-2 compound-Gaussian component was modeled as the scattering arising from two independent, unit variance Rayleigh distributed discrete reflectors excited by a monochromatic signal pulse located $\pm 1/2$ DFT bin in wavenumber space symmetrically about broadside modulated by the square-root of a Gamma random variable with a shape parameter of .1. The white noise variance, σ^2 was set to .1.

A total of 24 signal-free data snapshots were used for the secondary or training data set. The primary data set for detection consisted of a single data snapshot. The signal direction of arrival was chosen to be broadside to the array. The signal power to interference ratio (SIR) is defined as $10 log_{10} \sigma^2$. 15000 independent trials with and without a signal injected were performed, computer simulating the new GLRT, optimum MSD, ASD, and Kelly's CFAR GLRT [7] receivers. For comparison, an analogous pure Gaussian noise case with the same nominal covariance matrix was also simulated. Note that the ASD was implemented by using the 24 snapshot signal-free secondary data set to estimate the rank-2 interference subspace via a SVD and plugging the estimated noise subspace into (2).

Figures 2 and 3 show the empirically measured probability of detection (pd) curves obtained for a probability of false alarm (pfa) of .005 for the non-Gaussian and Gaussian cases respectively for all four detectors. From the pd curves in figure 2, it can be seen that the new GLRT has nearly the same performance as the optimum MSD and significantly outperforms the ASD and Kelly's GLRT when the interference is compound-Gaussian. However, it is interesting to observe that for the pure Gaussian case (figure 3), both the new GLRT and the ASD perform nearly as well as the optimum MSD.

4.1. Insensitivity to Rank Mismatch

An important question is the sensitivity of the detector to subspace model order or rank errors. The detector needs knowledge of the subspace model order of the interference. Since this information is usually not available, it must be estimated from the data. Thus the assumed rank or order may be erroneous and it is important to examine the sensitivity of the detector to incorrect rank. Using the same rank-2 interference example as before, we evaluated the new GLRT performance for assumed ranks varied between 1-5 on the basis of ROC curves measured from 2000 simulations and plotted them in figure 4. When the rank was underdetermined, that is, set to 1, the detector probability of detection was greatly reduced. This behavior is expected since we are allowing strong coherent interference to be treated as white noise. As the rank was over-determined between 3-5, the detector performance only degraded slowly as the rank was increased and appeared to be relatively insensitive to rank over-determination.

5. CONCLUSION

We have derived a new GLRT detector and shown its relationship to the UMPI MSD. Our perturbation analysis and numerical examples suggest that the new GLRT is likely to be much more robust in low rank non-Gaussian clutter than *ad hoc* or conventional adaptive detectors. Finally, further work needs to done in analyzing the detectors performance in regards to signal and rank mismatch and higher-order effects due to the non-Gaussianity of the interference.

APPENDIX A: PERTURBATION ANALYSIS

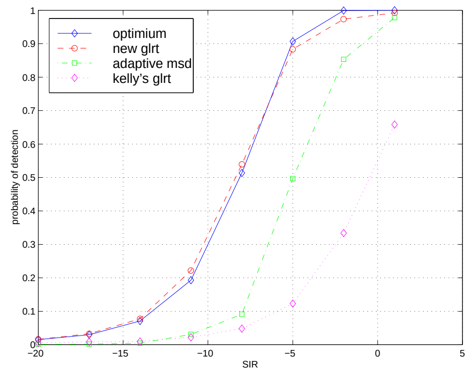

Figure 2: Experimentally measured probability of detection in non-Gaussian interference as a function of SIR for a pfa of .005 based on 15000 trials.

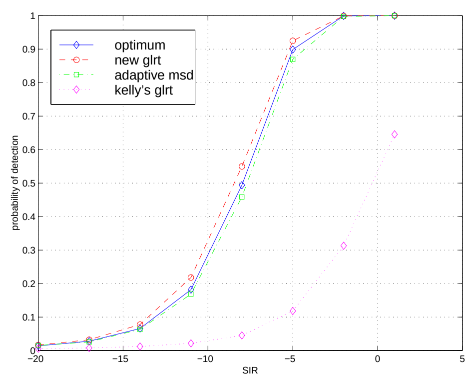

Figure 3: Experimentally measured probability of detection in Gaussian interference as a function of SIR for a pfa of .005 based on 15000 trials.

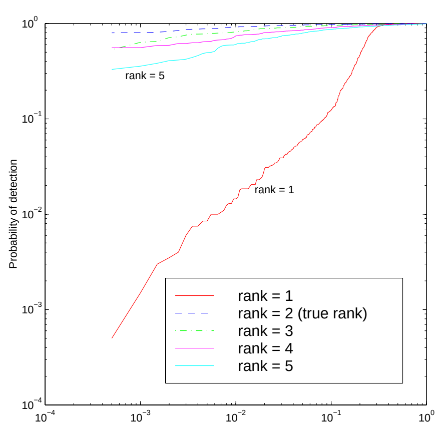

Figure 4: Experimentally measured ROC curves for the new GLRT detector when incorrect ranks are assumed for the subspace interference component.

We start with the numerator of (9). Recall that the numerator is the square-error in fitting a rank r_n matrix to Z. Letting $Z = AB + N$, where N is some perturbation and using the first-order subspace perturbation expansion derived in [6] for the error in approximating a matrix by a matrix of lower rank, we obtain

$$\min_{B,A} \|Z - BA\|_F^2 \approx \widehat{num} = \|P_B^{\perp} Z P_A^{\perp}\|_F^2 \tag{12}$$

where $P_A^{\perp} = I - A^H(AA^H)^{-1}A$.

We now approximate the denominator. If the denominator of (9) is solved with respect to only B_1 and A_1 (holding c fixed), it is equivalent to finding the rank r_n approximation to $Z - [0|Sc]$. Treating $[0|Sc]$ as a perturbation (weak signal and noise case) initially and applying (12), we can approximate the denominator as

$$\widehat{den} \approx \min_{c} \|P_B^{\perp} Z P_A^{\perp} - cP_B^{\perp}[0|S]P_A^{\perp}\|_F^2 \tag{13}$$

The minimization of (13) is a standard linear least-squares problem and the residual fitting error is

$$\widehat{den} \;\approx\; \|P_{S''}^{\perp}\mathbf{z}''\|_F^2 \tag{14}$$

where $\mathbf{z}'' = vec(P_B^{\perp} Z P_A^{\perp})$, $P_{S''}^{\perp} = I - P_{S''}$, $P_{S''} \;=\; S''({S''}^H S'')^{-1}S''$, and $S'' \;=\; [vec(P_B^{\perp}[0|S]P_A^{\perp})]$. The operator $vec(\cdot)$ takes a matrix and converts it to a vector representation by stacking the columns. Finally, replacing the exact quantities in (9) by their above approximations (12) and (14), and after some simplification, we obtain

$$y \approx 1 + \frac{\| P_{S''}\ \mathbf{z}''\|_F^2}{\| P_{S''}^{\perp}\ \mathbf{z}''\|_F^2} = \frac{\widehat{num}}{\widehat{den}} \tag{15}$$

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