TURBO-EQUALIZATION: CONVERGENCE ANALYSIS

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ABSTRACT

We investigate a sub-optimal iterative receiver for joint equalization and decoding called Turbo-equalizer. We view the evolution of the error variance of the transmitted symbols through the iterative processing, obtaining convergence analysis. This allows us to predict the asymptotic performance (when the Turbo-equalizer has converged) but also the trigger point observed in its performance.

1. INTRODUCTION

In high rate communication, where the transmitted signal is subject to intersymbol interference (ISI), we may use equalization to reduce the effect of ISI and channel coding to correct remaining errors. A conventional equalizer does not make use of the redundancy introduced by the channel coding. Equalization and decoding are disjoint which is not optimal in the sense of the minimization of the error probability. Since optimal joint equalization and decoding is an NP-complete problem, we consider a relevant trade-off between complexity and performance: the Turbo-equalizer, first proposed in [4] and studied in [7]. Our goal is to analyze the evolution of the effective error variance through the iterative processing involved in the Turbo-equalizer, following the approach described in [1]. This may allow prediction of the performance of the Turbo-equalizer, without a need to run the iterative processing.

2. TURBO-EQUALIZER

Consider the transmitter described in Figure 1. The discrete channel is characterized by its impulse response h_n. The samples of the received signal can be written as $r_n = h \star d_n + w_n$, where $\star$ stands for the convolution and w_n is a white gaussian noise.

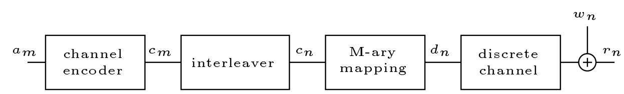


Figure 1: Transmitter.

The Turbo-Equalizer is implemented in a modular pipelined structure with P identical iterations. With reference to Figure 2, each module consists of the concatenation of a Soft Input/Soft Output equalizer, a soft de-mapper (symbol to bit), a deinterleaver, a *Maximum a posteriori* (MAP) decoder, an interleaver and a soft re-mapper (bit to symbol). The output of iteration p together with the channel output is input to iteration $p+1$. Decoding is carried out by using the MAP algorithm [2]. After deinterleaving, an estimate d_n^p of the conditional mean value of the symbols d_n is calculated as in [4]. In the case of binary modulation (BPSK): $d_n^p = \mathcal{P}(c_m = 1 \mid \mathbf{y}) - \mathcal{P}(c_m = 0 \mid \mathbf{y})$.

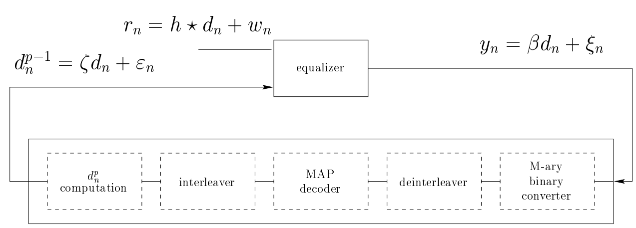


Figure 2: Module p of the Turbo-equalizer.

As shown in Figure 3, the equalizer used here is the Interference Canceler (IC) proposed in [4]. It consists of two filters $P(z)$ and $Q(z)$ and is fed by both the channel output, r_n and the output of the previous module, d_n^{p-1}. Note that the equalizer of the very first iteration is a Decision Feedback Equalizer (DFE) that just processes the output of the channel, since d_n^0 does not exist. Minimization of the mean square error $\text{MSE} = E[\mid y_n - d_n \mid^2]$ over the coefficients of the filters P and Q under the constraint $q_0 = 0$, when Q is fed by

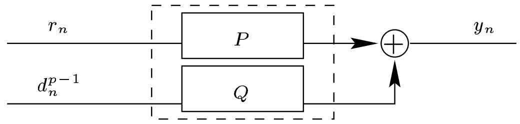


Figure 3: Equalizer in the module $p > 1$.

the d_n, yields the filters in the form [4]:
$P(z) = \alpha H^*(z^{-1*})$, $Q(z) = \alpha(H(z)H^*(z^{-1*}) - \gamma_h(0))$
where $H(z) = \sum_i h_i z^{-i}$, $H^*(z^{-1*}) = \sum_i h_i^* z^i$,
$\gamma_h(0) = \sum_i |h_i|^2$ and $\alpha = \frac{\sigma_d^2}{\sigma_d^2 \gamma_h(0) + \sigma_w^2}$.
* stands for conjugation whereas σ_d^2 and σ_w^2 stand respectively for the transmitted symbol power and thermal noise power. P is the filter matched to the channel H and $Q + \alpha\gamma_h(0)$ its autocorrelation. Q is used to remove the ISI caused by previous and future detected symbols. We have shown earlier [7] that it results in *complete* elimination of ISI, provided that the previous and future decisions are correct.

3. CONVERGENCE ANALYSIS

Note that the Turbo-equalizer presented above gives an estimate of the transmitted symbol d_n in two places: the equalizer output y_n and decoder output d_n^p. Let us now split the Turbo-equalizer in two blocks and write the input and output of each block explicitly as estimates of d_n. This leads to the scheme shown in Figure 2. The errors ε_n and ξ_n contain both remaining ISI and noise respectively at the input and output of the equalizer. For tractable analysis, we follow the approach described in [1] and represent the Turbo iteration as the evolution of error variances on d_n.

3.1. Model and principle

The Turbo-equalizer is fed by the output of the channel:
$$r_n = h \star d_n + w_n,$$
with normalized thermal noise variance $\sigma_{w,N}^2 = \frac{\sigma_w^2}{\gamma_h(0)\sigma_d^2}$.
The estimates of d_n are either:
$$d_n^{p-1} = \zeta d_n + \varepsilon_n,$$
with effective normalized error variance $\sigma_{\varepsilon,N}^2 = \frac{\sigma_\varepsilon^2}{\zeta^2 \sigma_d^2}$, or
$$y_n = \beta d_n + \xi_n,$$
with effective normalized error variance $\sigma_{\xi,N}^2 = \frac{\sigma_\xi^2}{\beta^2 \sigma_d^2}$, where $_N$ stands for normalized.

Considering input variances $\sigma_{w,N}^2$ and $\sigma_{\varepsilon,N}^2$ to the IC block, we compute the output error variance $\sigma_{\xi,N}^2$. Under the assumptions that $\zeta = 1$ and that w_n and ε_n are independent,

$$\sigma_{\xi,N}^2 = g_{\sigma_{w,N}^2}\left(\sigma_{\varepsilon,N}^2\right) = \sigma_{w,N}^2 + \frac{\Gamma_h}{\gamma_h(0)^2}\sigma_{\varepsilon,N}^2,$$

where Γ_h is obtained from the autocorrelation of the autocorrelation $\gamma_h(n)$ of the channel h_n where the central term is suppressed:

$$\frac{\Gamma_h}{\gamma_h(0)^2} = \frac{1}{\gamma_h(0)^2}\left(\sum_n \gamma_h(n)\gamma_h(-n)^*\right) - 1, \qquad (1)$$

which is a measure of the channel dispersion. Note that the larger the slope of $g_{\sigma_{w,N}^2}$ is, the larger the output variance is and the tougher the channel is. So we can define a "tough" channel, when processed by the Turbo-equalizer, as a channel with large $\frac{\Gamma_h}{\gamma_h(0)^2}$.

The decoder updates the error variance $\sigma_{\varepsilon,N}^2$ via the function f:
$$\sigma_{\varepsilon,N}^2 = f\left(\sigma_{\xi,N}^2\right).$$
f may be obtained through simulation, or bounded. Analytical characterization of f is difficult and we focus on understanding the Turbo iteration, given f via simulation over the AWGN channel.

We can now test the Turbo-equalizer convergence by plotting the output variance of the decoder $\sigma_{\varepsilon,N}^2$ versus the input variance $\sigma_{\xi,N}^2$ (that is to say f) and the input variance of the equalizer (IC) $\sigma_{\varepsilon,N}^2$ versus the output variance $\sigma_{\xi,N}^2$ for a given thermal noise variance $\sigma_{w,N}^2$ (that is to say $g_{\sigma_{w,N}^2}^{-1}$). One Turbo iteration corresponds to the recurrence:

$$\sigma_{\varepsilon,N}^{2\ p+1} = f\left(\sigma_{\xi,N}^{2\ p+1}\right) = f \circ g_{\sigma_{w,N}^2}\left(\sigma_{\varepsilon,N}^{2\ p}\right).$$

Fixed points of $f \circ g_{\sigma_{w,N}^2}$ and their stability represent the asymptotic convergence points of the processing. Given a fixed point x, the condition for stability is:

$$\left|\left(f \circ g_{\sigma_{w,N}^2}\right)'(x)\right| < 1,$$

which depends indeed on $\sigma_{w,N}^2$.

Note that, when convergence of the Turbo-equalizer is achieved and under gaussian assumption of the different errors, the variance output of the decoder of the fixed point can be easily related to the performance in terms of Bit error rate. So, this allows to predict the final performance of the Turbo-equalizer. We are now interested in the analysis of the performance.

3.2. "Easy" channels

Let us consider "easy" channels, *i.e.* channels with small dispersion coefficient (1), for instance Porat and Friedlander's channel [5] with dispersion 0.73 in Figure 4. It also corresponds to a channel with the same minimal distance as the AWGN one. We use here a 64-state recursive systematic code [133,171].

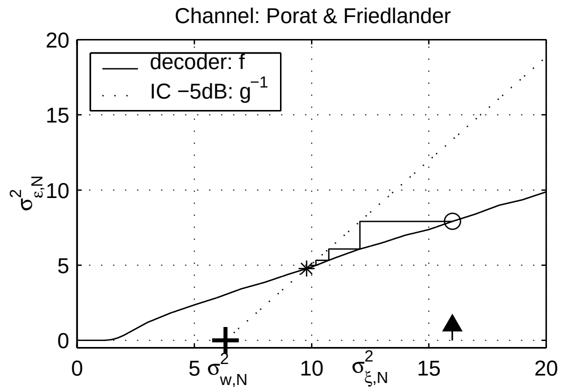


Figure 4: Iterative process of the Turbo-equalizer (Porat and Friedlander's channel). Starting the first iteration at the arrow setting.

In practice, we have observed the existence of a stable fixed point for these "easy" channels. Moreover simulations show that the Turbo-equalizer tends to the performance of the coded sequence transmitted over AWGN channel at high SNR but not at low SNR [7]. This can be easily explained with the convergence analysis (see Figure 5). Given a noise variance $\sigma^2_{w,N}$, the decoder gives an output error variance plotted by $+$. As for the Turbo-equalizer, it tends to the fixed point $*$, which leads to an extra variance Δ for the Turbo-equalizer at high σ^2_w.

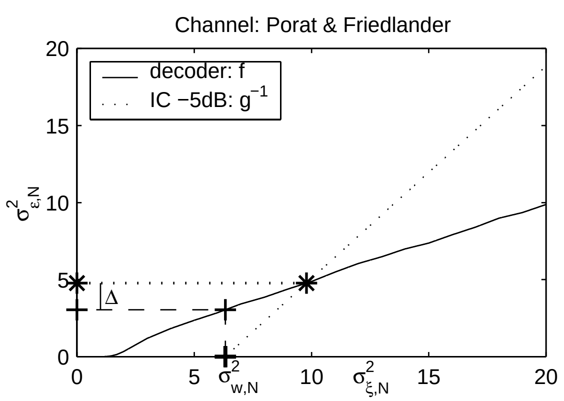


Figure 5: Convergence analysis.

3.3. "Tough" channels

In this section, we consider "tough" channels (Proakis B and C [6, page 616]) such as the coefficient (1) of which is respectively 0.94 and 2.06. It also corresponds to channels with smaller minimal distance than the AWGN one. For these channels, the characteristic of the decoder during the Turbo simulation differs from the function f simulated above for an AWGN channel. In spite of this mis-matched decoding, $\sigma^2_{\varepsilon,N}$ may be further used to predict the performance of the Turbo-

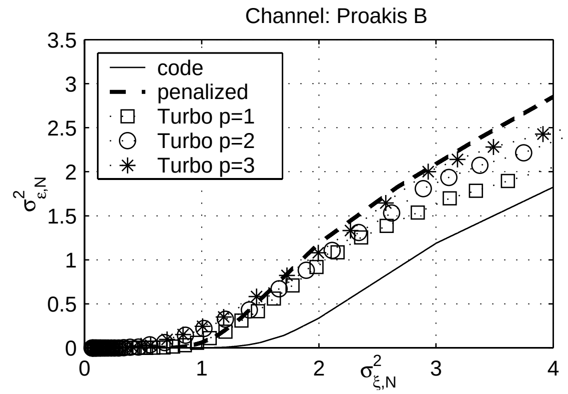


Figure 6: Accuracy of the penalized curve of the decoder and the simulated ones.

equalizer (without carrying out the simulation). We propose to penalize the input variance of the decoder, $\sigma^2_{\xi,N}$, with the ratio between the minimal distances of the dispersive and of the AWGN channel, which defines the channel loss, a:

$$\sigma^2_{\varepsilon,N} = f\left(\sigma^2_{\xi,N}\ \underbrace{\frac{d^2_{\text{min dispersive channel}}}{d^2_{\text{min AWGN channel}}}}_{a}\right),\ \text{where } a \leq 1.$$

The accuracy of the prediction depends on how the penalized function matches the simulated ones (see Figure 6).

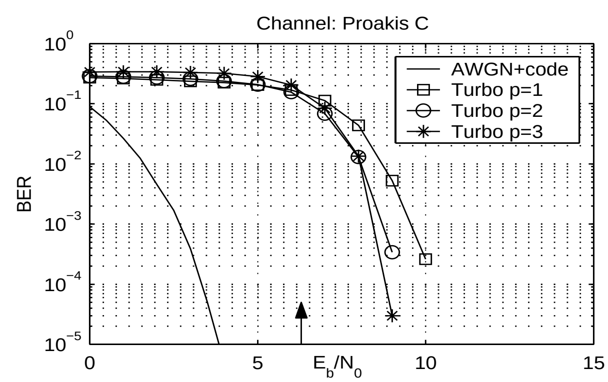
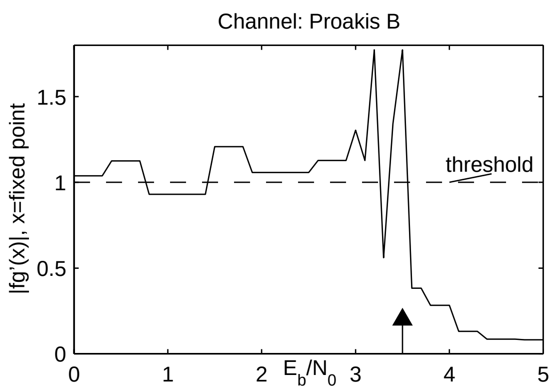


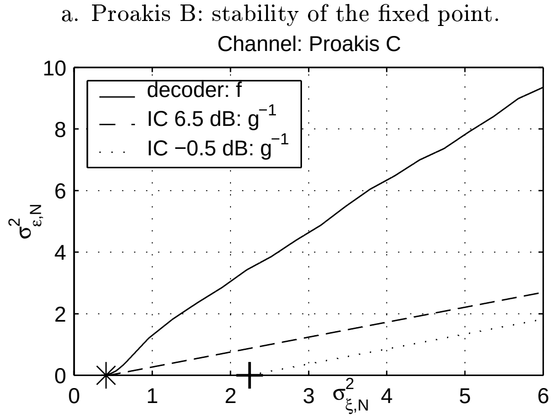
Figure 7: Simulated performance of the Turbo-equalizer: trigger point at 6 dB with MAP equalizer at the 1^{st} iteration.

We have shown earlier ([7] and Figure 7) that there is a trigger point in the iterative process, followed by a breakdown effect. After the trigger point, the BER decreases steeply as a function of the decoding step p. As we run the iterative process and plot the results in terms of error variances, we observe that the trigger point corresponds to the limit of convergence to a fixed point. In the following, we use our analysis in order to predict this trigger point.

The analysis for the Proakis B channel shows that there is a limit of the stability of the fixed point that may be related to the trigger point (1.5 to 2 dB for simulation to be compared with 3 dB for analysis, see Figure 8.a). Also shown is the prediction of the trigger point for channel Proakis C. For this channel, note that the fixed point does not always exist, depending on $\sigma^2_{w,N}$. The limit of existence of this fixed point occurs at 6.5 dB as is shown in Figure 8.b and matches reasonably well with the trigger point (6 to 10 dB for simulation, depending on the equalizer of the first iteration). Note that when the fixed point doesn't exist, the slope of $f \circ g_{\sigma^2_{w,N}}$ is greater than 1 and the output variance after one iteration is greater than the input variance. So, before the trigger point, BER increases as a function of the decoding step p as is shown in Figure 7.



a. Proakis B: stability of the fixed point.



b. Proakis C: existence of the fixed point.

Figure 8: Prediction of the trigger point.

4. CONCLUSION

We analyzed the error variances and the evolution of these variances through the Turbo-equalizer, obtaining a convergence analysis. Because of mis-matched decoding during the iterative process, we had to penalize the decoder with the ratio between the minimal distances of the dispersive and of the AWGN channel. This allowed us to predict the trigger point observed in Turbo-equalizer's performance without having to run the complete simulation. Depending on the channel, the prediction is based on either the limit of existence of the fixed point or the limit of stability of this point (if the fixed point exists). Based on this analysis, we propose a definition of a "tough" channel, when processed by the Turbo-equalizer.

The analysis of the Turbo-equalizer, we just proposed, is complete when the distribution of the estimates of the transmitted symbol d_n (given d_n) is a white gaussian one. In the tough cases, we observed on simulation that the noise at the output of the IC is white but not gaussian (using the D'Agostino's test [3] based on third and fourth order statistics). This may explain why the decoder's performance is reduced.

5. REFERENCES

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