MINIMUM MEAN SQUARE ERROR NONUNIFROM FIR FILTER BANKS

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ABSTRACT

Theory for jointly optimizing nonuniform analysis and synthesis FIR filter banks with arbitrary filter lengths and an arbitrary delay through the filter bank is developed. The FIR subband coder is optimized with respect to the minimum mean square error between the output and the input signals under a bit constraint. The subband quantizers are modeled as additive noise sources. Theoretical comparisons are made against a well-known 5_3 wavelets used in a tree-structure. The proposed filter banks, which are both rate and source dependent, have a better distortion rate performance. Equations for finding jointly optimized analysis and synthesis filter banks under a power constraint are also presented.

1. INTRODUCTION

Filter banks are widely used in source coding. Thus, it is important to optimize the filter banks. In this paper, the problem of jointly optimizing the analysis and synthesis FIR nonuniform filter banks is studied.

The subband coding model is shown in Figure 1. The objective is to minimize the mean square error (MSE) between the input and the output of the system under a bit constraint with respect to the bit allocation as well as the analysis and synthesis filter banks.

For a given uniform analysis FIR filter bank, the optimal uniform FIR synthesis filter bank has been derived in various ways [1, 2, 3] in the case where all the synthesis filters are of the same length and the overall filter bank delay can be expressed as $kM-1$ where k is a positive integer and M is the decimation factor. These synthesis filter banks can also be called uniform FIR Wiener filter banks. In [3], the joint optimization of the uniform synthesis filter bank and the bit allocation was studied. The first stage of a tree-structured synthesis filter bank was optimized in [4]. Jointly optimized analysis and synthesis bit constrained uniform FIR filter banks were treated in [5]. For unknown input signals, the problem of optimizing the nonuniform synthesis filter bank was studied in [6].

This paper was inspired by the work described in [7], and the same way of treating the non-stationarity introduced by the decimators/expanders is used here. The nonuniform case has not been studied in details, and one of the contributions of this paper is to derive the Wiener filter bank for nonuniform FIR filter banks.

This rest of this paper is organized as follows: The assumptions and the problem under consideration are stated in Section 2. In Section 3, the equations required for the optimization are discussed and a technique for optimizing nonuniform filter banks with arbitrary filter lengths is described. Results and comparisons with other filter banks are given in Section 4, while conclusions are drawn in Section 5.

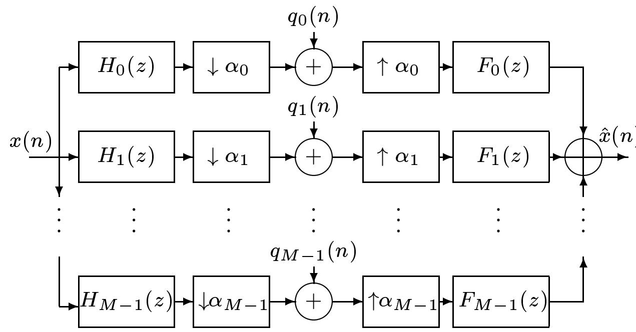

Fig. 1. Nonuniform filter bank system model.

2. PROBLEM FORMULATION

Figure 1 shows the system under consideration. The delay through the system is Δ. The maximum number of quantizers receiving a positive number of bits is M_{max}, but the number of quantizers actually receiving a positive number of bits is denoted by M. Obviously, $0 \leq M \leq M_{\text{max}}$, and M has to be found through a discrete optimization.

Let the nonuniform integer decimation factors be denoted by α_i, and let the least common multiplier of the M factors that are used be denoted by τ, which in general depends on M. The $M \times M$ diagonal matrix $\boldsymbol{\alpha}$ is defined to have the element α_i as its diagonal element number i. The decimation factors do not necessarily have to result in critically decimated filter banks, because the theory developed here is general including the critically decimated case. Systems having so-called block decimation and expansion [7] are also included in the theory. Since the filter bank system is linear periodically time varying with period τ [7], this has to be taken into account when finding the performance expressions for the system.

Let the following $M \times N$ matrix contain the impulse responses of the synthesis filter bank:

$$\boldsymbol{R} = [\boldsymbol{r}(0)|\boldsymbol{r}(1)|\dots|\boldsymbol{r}(N-1)], \tag{1}$$

where $\boldsymbol{r}(k)$ is an $M \times 1$ vector. Row number k in the matrix $\boldsymbol{R}$ contains the synthesis impulse response number k. The numbering of the indices start at zero in the first column of $\boldsymbol{R}$. Similarly, let the $M \times N$ matrix $\boldsymbol{E}$ contain the impulse responses of the analysis filters $H_i(z)$.

2.1. Decimators and Expanders

The combination of the decimators and expanders can be treated by multiplying the $M \times 1$ vectors containing the subband samples

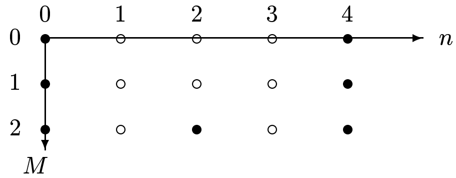

Fig. 2. Time frequency lattice for $(\alpha_0, \alpha_1, \alpha_2) = (4, 4, 2)$.

with periodic time varying diagonal matrices [7]. These $M \times M$ matrices are denoted by $\boldsymbol{\Lambda}(i)$.

Figure 2 shows a time frequency lattice of how one period of the matrices $\boldsymbol{\Lambda}(i)$ can be found. In the figure, n is the time index and M is the number of subbands that are used in the subband coder. The decimation factors are given by $(\alpha_0, \alpha_1, \alpha_2) = (4, 4, 2)$ in the example shown in the figure. From the figure, it is seen that the $M \times M$ matrices taking care of the cyclo wide sense stationarity are given by:

$$\begin{aligned} \boldsymbol{\Lambda}(0) &= \boldsymbol{I}, \\ \boldsymbol{\Lambda}(1) &= \boldsymbol{\Lambda}(3) = \boldsymbol{0}, \\ \boldsymbol{\Lambda}(2) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{2}$$

The noise samples that are added to the subband samples between the decimators and expanders can be moved *in front of* the decimators without altering the input-output relationship of the system. The reason for this is that exactly the same quantization samples are found after the expanders in both cases.

The diagonal matrix $\boldsymbol{\Lambda}(i)$ is periodic with period τ, i.e.,

$$\boldsymbol{\Lambda}(i+\tau) = \boldsymbol{\Lambda}(i). \tag{3}$$

Define the diagonal matrix $\boldsymbol{\Lambda}$ as the following sum:

$$\boldsymbol{\Lambda} = \sum_{i=0}^{\tau-1} \boldsymbol{\Lambda}(i). \tag{4}$$

It can be shown that $\boldsymbol{\Lambda} = \tau \boldsymbol{\alpha}^{-1}$.

2.2. Correlation Matrices and Quantization Model

An additive signal-dependent colored quantization noise model is assumed. Correlation matrices that are used in order to express the performance of the system are introduced next.

The following $N \times N$ and $M \times M$ autocorrelation matrices are defined as

$$\begin{aligned} \boldsymbol{\Phi}_{\boldsymbol{x}}^{(N)}(l) &= E[\boldsymbol{x}(l+n)\boldsymbol{x}^T(n)], \\ \boldsymbol{\Phi}_{\boldsymbol{u}}^{(M)}(l) &= E[\boldsymbol{u}(l+n)\boldsymbol{u}^T(n)], \end{aligned} \tag{5}$$

where the $N \times 1$ vector $\boldsymbol{x}(n)$ and the $M \times 1$ vector $\boldsymbol{u}(n)$ are defined as:

$$\begin{aligned} \boldsymbol{x}(n) &= [x(n), x(n-1), \dots, x(n-(N-1))]^T, \\ \boldsymbol{u}(n) &= [u_0(n), u_1(n), \dots, u_{M-1}(n)]^T. \end{aligned} \tag{6}$$

Here, the input time series of the filter banks is denoted by $x(n)$ and the additive quantization noise after analysis filter number i and before decimating by α_i is denoted by $u_i(n)$.

An $N \times 1$ autocorrelation and an $M \times 1$ cross-correlation vector are also needed. These vectors are defined as

$$\boldsymbol{\phi}_{\boldsymbol{x}}^{(N)}(l) = E[\boldsymbol{x}(n+l)x(n)], \tag{7}$$

$$\boldsymbol{\phi}_{\boldsymbol{u},x}^{(M)}(l) = E[\boldsymbol{u}(n+l)x(n)]. \tag{8}$$

In addition, the following $N \times M$ cross-correlation matrix:

$$\boldsymbol{\Phi}_{\boldsymbol{x},\boldsymbol{u}}^{(N,M)}(l) = E[\boldsymbol{x}(l+n)\boldsymbol{u}^T(n)], \tag{9}$$

is used.

If high rates are assumed, the variance of the noise in quantizer number $i \in \{0, 1, \dots, M-1\}$ can be modeled as [8]

$$\sigma_{q_i}^2 = c_i \sigma_{y_i}^2 2^{-2b_i}, \tag{10}$$

where $\sigma_{y_i}^2$ is the variance of the corresponding subband signal $y_i(n)$ and c_i is the coding coefficient that depends on the coding technique and the probability density function of the subband signal. Since b_i, the number of bits used in quantizer number i, is decided by the relationship between $\sigma_{y_i}^2$ and $\sigma_{q_i}^2$, the choice $\sigma_{q_i}^2 = 1$ can be made without loss of generality, provided that c_i is known.

2.3. MSE and Bit Expressions

By expressing the output of the synthesis filter bank $\hat{x}(n)$ in the same way as in [7], and also including the additive quantization noise it can be shown that the MSE per source sample $\epsilon_{N,M}(\Delta)$ can be written as:

$$\begin{aligned} \epsilon_{N,M}(\Delta) &= \frac{1}{\tau} \sum_{i=0}^{\tau-1} E\left[(\hat{x}(n+i) - x(n+i-\Delta))^2\right] \\ &= \sigma_x^2 + \frac{1}{\tau} \Bigg\{ \sum_{k=0}^{N-1} \sum_{p=0}^{N-1} \boldsymbol{r}^T(k) \sum_{i=0}^{\tau-1} \boldsymbol{\Lambda}(i) \Big[\boldsymbol{E}\boldsymbol{\Phi}_{\boldsymbol{x}}^{(N)}(p-k)\boldsymbol{E}^T \\ &\quad + \boldsymbol{\Phi}_{\boldsymbol{u}}^{(M)}(p-k) + \boldsymbol{E}\boldsymbol{\Phi}_{\boldsymbol{x},\boldsymbol{u}}^{(N,M)}(p-k) \\ &\quad + \left(\boldsymbol{E}\boldsymbol{\Phi}_{\boldsymbol{x},\boldsymbol{u}}^{(N,M)}(k-p)\right)^T \Big] \boldsymbol{\Lambda}(i-(p-k))\boldsymbol{r}(p) \\ &\quad -2 \sum_{k=0}^{N-1} \boldsymbol{r}^T(k) \boldsymbol{\Lambda} \left(\boldsymbol{E}\boldsymbol{\phi}_{\boldsymbol{x}}^{(N)}(\Delta-k) + \boldsymbol{\phi}_{\boldsymbol{u},x}^{(M)}(\Delta-k) \right) \Bigg\}, \end{aligned} \tag{11}$$

where σ_x^2 is the variance of the input time series $x(n)$, which is assumed to be wide sense stationary and having zero mean.

By using Equation (10), the bit constraint $\sum_{i=0}^{M-1} \frac{b_i}{\alpha_i} = b$ can be expressed as

$$\sum_{i=0}^{M-1} \frac{\ln \sigma_{y_i}^2}{\alpha_i} = \ln \beta, \tag{12}$$

where $\beta = 2^{2b} \prod_{i=0}^{M-1} \left(\frac{\sigma_{q_i}^2}{c_i}\right)^{\frac{1}{\alpha_i}}$ is a constant.

The problem is to find the optimal values of the bit distribution as well as analysis and synthesis filter banks which minimize the MSE given by Equation (11) subject to the bit constraint of Equation (12). Since the choices $\sigma_{q_i}^2 = 1$ have been made, the optimal bit distribution is decided when the subband variances $\sigma_{y_i}^2$ are known. These are known when the analysis filter bank is found. Therefore, the optimization of the analysis filter bank also indirectly finds the optimal bit distribution.

3. OPTIMIZATION ALGORITHM

In order to derive the equations for finding the jointly optimized analysis and synthesis filter banks, the operator vec is used. This operator places the columns of a matrix into a long vector where the first column is placed in the top of this vector. The constrained optimization problem is converted to an unconstrained problem by means of the Lagrange multiplier method. Then necessary conditions for optimality are found by differentiating the unconstrained objective function with respect to the unknown parameters. When deriving the optimization equations, results from [9] are used.

3.1. Equal Filter Lengths

It can be shown that for a fixed analysis filter bank the equations for the optimal synthesis filter bank having equal filter lengths can be expressed as

$$\text{vec}(\boldsymbol{R}) = \boldsymbol{A}^{-1}\boldsymbol{D}, \tag{13}$$

where the matrix $\boldsymbol{A}$ is a symmetric $NM \times NM$ block Toeplitz matrix. Therefore, the whole matrix $\boldsymbol{A}$ can be found from the first block row. The block element number l of dimension $M \times M$ in the first row is given by

$$\boldsymbol{a}_l = \sum_{i=0}^{\tau-1} \boldsymbol{\Lambda}(i+l)\left(\boldsymbol{E}\boldsymbol{\Phi}_{\boldsymbol{x}}^{(N)}(l)\boldsymbol{E}^T + \boldsymbol{\Phi}_{\boldsymbol{u}}^{(M)}(l) + \boldsymbol{E}\boldsymbol{\Phi}_{\boldsymbol{x},\boldsymbol{u}}^{(N,M)}(l) + \left(\boldsymbol{E}\,\boldsymbol{\Phi}_{\boldsymbol{x},\boldsymbol{u}}^{(N,M)}(-l)\right)^T\right)\boldsymbol{\Lambda}(i), \tag{14}$$

where $l \in \{0, 1, \dots, N-1\}$. The $NM \times 1$ vector $\boldsymbol{D}$ can be expressed in terms of the sub-vectors of dimension $M \times 1$, and they are denoted by $\boldsymbol{d}_l$, where $l \in \{0, 1, \dots, N-1\}$. These sub-vectors can be expressed as

$$\boldsymbol{d}_l = \boldsymbol{\Lambda}\left(\boldsymbol{E}\boldsymbol{\phi}_{\boldsymbol{x}}^{(N)}(\Delta - l) + \boldsymbol{\phi}_{\boldsymbol{u},x}^{(M)}(\Delta - l)\right). \tag{15}$$

It can be shown that for a fixed synthesis filter bank the equations for the optimal analysis filter bank having equal filter lengths can be written as

$$\text{vec}(\boldsymbol{E}) = \left[\sum_{k=0}^{N-1}\sum_{p=0}^{N-1} \boldsymbol{\Phi}_{\boldsymbol{x}}^{(N)}(p-k) \otimes \boldsymbol{C}_{k,p} + \boldsymbol{\Phi}_{\boldsymbol{x}}^{(N)}(0) \otimes \left(\mu\boldsymbol{\Lambda}\boldsymbol{\Sigma}_{\boldsymbol{y}}^{-1}\right)\right]^{-1} \text{vec}\left(\boldsymbol{\Lambda}\boldsymbol{R}\boldsymbol{J}\boldsymbol{\Phi}_{\boldsymbol{x}}^{(N)}(N-1-\Delta) - \sum_{k=0}^{N-1}\sum_{p=0}^{N-1} \boldsymbol{C}_{k,p}\left(\boldsymbol{\Phi}_{\boldsymbol{x},\boldsymbol{u}}^{(N,M)}(p-k)\right)^T\right), \tag{16}$$

where the $N \times N$ matrix $\boldsymbol{J}$ is the counter identity matrix and the operator $\otimes$ is the Krönecker product. The $M \times M$ matrix $\boldsymbol{C}_{k,p}$ is given by

$$\boldsymbol{C}_{k,p} = \sum_{i=0}^{\tau-1} \boldsymbol{\Lambda}(i)\boldsymbol{r}(k)\boldsymbol{r}^T(p)\boldsymbol{\Lambda}(i-(p-k)). \tag{17}$$

The matrix $\boldsymbol{\Sigma}_{\boldsymbol{y}}$ is an $M \times M$ diagonal matrix containing the sub-band variance $\sigma_{y_i}^2$ as diagonal element number i. μ is a Lagrange multiplier for the bit constraint in Equation (12). Equation (16) is nonlinear because the matrix $\boldsymbol{\Sigma}_{\boldsymbol{y}}$ depends in the analysis matrix $\boldsymbol{E}$. This is the only term on the right hand side that depends on the analysis filter bank. Equation (16) can be solved by fix-point iteration [10].

An alternative constraint used in communication problems is the power constraint. This constraint can be expressed for nonuniform filter banks as:

$$\sum_{i=0}^{\tau-1} \frac{\sigma_{y_i}^2}{\alpha_i} = P, \tag{18}$$

where P is the average power used per source sample. If this constraint is used instead of the bit constraint in the optimization, then the synthesis filter bank equations are unchanged, but the analysis filter bank equations are the same except that the matrix $\boldsymbol{\Sigma}_{\boldsymbol{y}}$ becomes the identity matrix.

3.2. Arbitrary Filter Length Optimization

Theory for jointly optimized FIR analysis and synthesis filter banks with analysis and synthesis filter lengths N was developed earlier in this section. Here, this theory is extended to include the case where the filters can have *arbitrary given filter lengths*.

Row number i from *left* to *right* in the matrices $\boldsymbol{E}$ and $\boldsymbol{R}$ represents the impulse response of analysis filter number i, $H_i(z)$, and synthesis filter number i, $F_i(z)$, respectively. Since the filter lengths in the filter banks are not necessarily equal, the matrices $\boldsymbol{E}$ and $\boldsymbol{R}$ may contain impulse response coefficients that are forced to be zero-valued.

Let $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{R}}$ be $M \times N$ matrices containing ones at the positions corresponding to where the analysis filter bank $\boldsymbol{E}$ and synthesis filter bank $\boldsymbol{R}$ contain free parameters and zeros where $\boldsymbol{E}$ and $\boldsymbol{R}$ must contain zeros, respectively.

By using Lagrange multipliers [11], it can be shown that the equations for finding jointly optimized analysis and synthesis filter banks with *arbitrary given filter lengths* can be found by picking out the equations from Equations (16) and (13), respectively, corresponding to the positions where vec($\boldsymbol{\mathcal{E}}$) and vec($\boldsymbol{\mathcal{R}}$) are different from zero. In addition, in the positions corresponding to where vec($\boldsymbol{\mathcal{E}}$) and vec($\boldsymbol{\mathcal{R}}$) are equal to zero, the old equations in these positions are replaced with equations stating that the corresponding filter coefficients are equal to zero. In this method, the fixed filter coefficients could be set to an *arbitrary constant value*, not only zero, and this can be done for any coefficients in the impulse response.

Since the matrices $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{R}}$ may contain zeros and ones at arbitrary positions, the above procedure can be used to find jointly optimized analysis and synthesis filter banks with arbitrary given filter lengths. This is done by choosing an appropriate shape of the matrices $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{R}}$. While choosing the shape of these matrices, it is important to remember that the delay through each branch of the analysis/synthesis filter bank combination must be the same if the filter banks are desired to have the perfect reconstruction (PR) property. At high rates, it is asymptotically optimal to have PR filter banks, so the structure of the matrices $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{R}}$ should be chosen carefully.

The theory developed in this section also includes the uniform case if the $\boldsymbol{\Lambda}(i)$ matrices are chosen properly. In this case, the same results are achieved as in the uniform case considered in [5]. This fact can be used as a verification of the formulas.

The optimization of the filter banks is performed by first choosing the initial values that are required. Then, Equations (13) and (16) are alternatingly solved until convergence is reached.

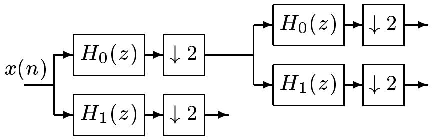

Fig. 3. Comparison are made to the following tree-structured filter bank. Only the analysis filter bank is shown.

4. RESULTS AND COMPARISONS

The PR filter banks used as an alternative solution is first introduced. Figure 3 shows the analysis filter bank of the PR system. The length of the filters $H_0(z)$ and $H_1(z)$ is 5 and 3, respectively, and the corresponding filters on the synthesis side have lengths 3 and 5, respectively. The filters are found from [12], and the bit allocation used is optimal bit allocation for PR filter banks [13] when using a white signal-independent quantization noise model. The optimal rate allocation is achieved when the product of the quantization noise variance and the squared norm of the synthesis filter in the same subband is constant for all subbands. This is actually the same criterion that is used for uniform filter banks.

With the decimation factors used in the tree-structure in Figure 3 the decimation factors found in the equivalent structure shown in Figure 1 are given by $\alpha_0 = 4$, $\alpha_1 = 4$, and $\alpha_2 = 2$. This means that $\tau = 4$. One period of the periodic diagonal matrices $\boldsymbol{\Lambda}(i)$ taking care of the decimation and expansion has already been given in Equation (2).

If the noble identities are used and the convolution of the filters shown in Figure 3 are calculated, then the structures of the matrices $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{R}}$ can be decided to be

$$\boldsymbol{\mathcal{E}} = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0&0&0&0&0 \end{bmatrix},$$
$$\boldsymbol{\mathcal{R}} = \begin{bmatrix} 1&1&1&1&1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&0&0 \\ 0&0&0&0&0&0&1&1&1&1&1&0&0 \end{bmatrix}. \tag{19}$$

This example was selected such that linear phase is possible in the optimized filter bank, since, in this case, the delay through each subband branch is the same.

It is assumed that a signal-independent white quantization noise model is used. This means that the matrix $\boldsymbol{\Phi}_{\boldsymbol{x},\boldsymbol{u}}^{(N,M)}(l)$ is a zero matrix and the matrix $\boldsymbol{\Phi}_{\boldsymbol{u}}^{(M)}(l)$ is diagonal. From the above matrices, it is seen that $N = 13$, $\Delta = 9$, and $M_{\text{max}} = 3$. It is assumed that entropy constrained scalar quantizers are used and that the input signal is a Gaussian AR(1) signal with correlation coefficient equal to 0.95, implying that $c_i = \frac{e\pi}{6}$ [8]. Figure 4 shows the performance of the PR system, the proposed optimized system, and the distortion rate function [8].

In order to improve the quantization model, the constraints $\sigma_{y_i}^2 \geq \sigma_{q_i}^2$ are included in the optimization when using the white uncorrelated noise model used in this example.

5. CONCLUSIONS

The problem of finding jointly optimized analysis and synthesis nonuniform FIR filter banks was studied. This problem was solved by finding equations for optimal analysis filter banks for a given synthesis filter bank and vice versa. From Figure 4, it is seen that the proposed filter banks outperform the PR filter bank for all rates.

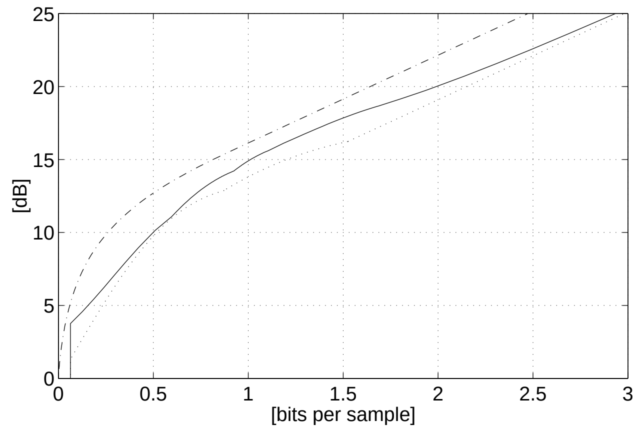

Fig. 4. Theoretical rate distortion performance of the proposed system (solid line), the PR system based on a tree-structure with the 5_3 filter bank (dotted line), and the distortion rate function (dash-dotted line). The parameters described in the text are used.

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