ISSUES OF FILTER DESIGN FOR BINARY WAVELET TRANSFORM

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ABSTRACT

Wavelet decomposition has recently been generalized to binary field in which the arithmetic is performed wholly in GF(2). In order to maintain an invertible binary wavelet transform with desirable multiresolution properties, the bandwidth, the perfect reconstruction and the vanishing moment constraints are placed on the binary filters. While they guarantee an invertible transform, the transform becomes non-orthogonal and non-biorthogonal in which the inverse filters could be signal length-dependent. We propose to apply the perpendicular constraint on the binary filters to make them length independent. A filter design strategy is outlined in which a filter design for a length of eight is given. We also propose an efficient implementation structure for the binary filters that saves memory space and reduces the computational complexity.

1. INTRODUCTION

Images in most applications are represented by a finite number of quantization levels such that the image ranges are finite. There have been several attempts to generalize wavelet decomposition to finite fields to take account of image characteristics [1-4]. In particular, [4] proposed a binary wavelet transform (BWT) in which the arithmetic is performed wholly in the GF(2), i.e., the field with binary elements, {0,1} and modulo-2 arithmetic. As the intermediate and the transformed data are binary, no quantization error is introduced. Modulo-2 arithmetic is equivalent to an exclusive-or operation; hence BWT can be performed efficiently.

The construction of a two-band BWT is equivalent to the design of a two-band perfect-reconstruction filter bank. In order to maintain an invertible transform with desirable multiresolution properties, three constraints [4] are placed on the binary filters. They are the bandwidth, the perfect reconstruction and the vanishing moment constraints. While these constraints guarantee an invertible BWT, they also confine the transform to be non-orthogonal and non-biorthogonal. We found that the form of the inverse filters might not be maintained when the signal length changes as in the up-sampling operation of the inverse transform. This is undesirable and extra constraints should be imposed so that the inverse filters are independent of the signal length.

We propose to use the perpendicular constraint to resolve this signal length-dependency problem. The perpendicular constraint relates the form of the forward and the inverse filters. A filter design is outlined using this constraint for a filter length of eight. In particular, a set of length-independent conditions is derived. An efficient in-place implementation structure is also explored which not only saves the memory space involved in the transform but also reduces the computational complexity.

2. LENGTH DEPENDENCY

The BWT is based on the circular convolution of binary sequences with binary filters (wavelet and the scaling function), and is followed by the process of decimation by two. Mathematically, an $N \times N$ transform matrix T is constructed as follows,

$$
\begin{aligned}
T &= \begin{bmatrix} C \\ D \end{bmatrix} \\
C &= \left(\bar{c}\big|_{s=0}, \bar{c}\big|_{s=2}, \cdots, \bar{c}\big|_{s=N-2} \right)' \\
D &= \left(\bar{d}\big|_{s=0}, \bar{d}\big|_{s=2}, \cdots, \bar{d}\big|_{s=N-2} \right)' \\
\bar{c} &= \left\{ c_0, c_1, c_2, c_3, \cdots, c_{N-2}, c_{N-1} \right\}' \\
\bar{d} &= \left\{ d_0, d_1, d_2, d_3, \cdots, d_{N-2}, d_{N-1} \right\}'
\end{aligned} \tag{1}
$$

$\bar{a}\big|_{s=k}$ defines a vector with elements formed from a circular shifted sequence of $\bar{a}$ by k, A' is the transpose of A, c_i and d_i are the scaling and the wavelet coefficients respectively. The BWT is then defined as,

$$y = Tx \tag{2}$$

where x is the original signal and y is the transformed signal. To guarantee an invertible BWT with multiresolution property, three constraints are required on the binary filters, namely the bandwidth, the perfect reconstruction and the vanishing moment constraints [4]. They are summarized as follows,

$$
\begin{aligned}
&\sum_{i=0,\,even}^{N-2} c_i = 0, \qquad \sum_{i=1,\,odd}^{N-1} c_i = 1 \\
&\sum_{i=0,\,even}^{N-2} d_i = 1, \qquad \sum_{i=1,\,odd}^{N-1} d_i = 1 \\
&d_{2i+1} = d_{2i} \qquad \text{for } 0 \le i < \frac{N}{2}
\end{aligned} \tag{3}
$$

Using eqn.3 and the fact that $a^2 = a$ for $a \in GF(2)$, the energy of the lowpass and the bandpass filters can be written as,

$$E_{low} = \sum_{i=0}^{N-1} c_i = 1 \qquad E_{band} = \sum_{i=0}^{N-1} d_i = 0 \tag{4}$$

In contrast to the real field, a vector in GF(2) can have zero energy even when the vector is not a zero vector. As the energy of the bandpass filter is zero, the BWT cannot be orthogonal or biorthogonal although it is invertible. The lack of orthogonality and biorthogonality renders the inverse filter design non-trivial.

Consider the filter coefficients given in [4] when the filter length is eight. The forward lowpass and bandpass filters are

$$\{1,1,1,0,1,0,1,0\}' \text{ and } \{1,1,1,1,1,1,0,0\}' \qquad (5)$$

respectively. When the signal length is 16, the forward BWT filters can be obtained simply by padding zeros to the end of the corresponding filters. However, the inverse filters are not related by this simple zero-padding relationship. In particular, for lengths that are equal to 8 and 16, the inverse lowpass filters are,

$$\{0,0,1,1,1,1,1,1\}',\ \{1,1,0,0,1,1,0,0,1,1,1,1,0,0,1,1\}'$$

and the inverse bandpass filters are,

$$\{0,1,0,1,0,1,1,1\}',\ \{0,0,1,0,0,0,1,0,0,1,1,1,0,1,0,1\}'$$

respectively. We refer to this as the signal length-dependency problem as the form of the inverse filters changes with signal length. Since the signal length changes at every decomposition level of BWT due to up-/down-sampling, this problem is very undesirable and an extra constraint is required on the binary filters.

3. PERPENDICULAR CONSTRAINT

While eqn.3 guarantees that the BWT is invertible, it does not explicitly specify the form of the inverse filters. The form of inverse filters is thus unconstrained such that it might change with signal length. Assume the inverse filter to have the form,

$$U = \begin{bmatrix} R \\ S \end{bmatrix}'$$

$$R = \left(\vec{r}\big|_{s=0}, \vec{r}\big|_{s=2}, \cdots, \vec{r}\big|_{s=N-2}\right)'$$

$$S = \left(\vec{s}\big|_{s=0}, \vec{s}\big|_{s=2}, \cdots, \vec{s}\big|_{s=N-2}\right)'$$

$$\vec{r} = \{r_0, r_1, r_2, r_3, \cdots, r_{N-2}, r_{N-1}\}'$$

$$\vec{s} = \{s_0, s_1, s_2, s_3, \cdots, s_{N-2}, s_{N-1}\}' \qquad (6)$$

This form should be independent of the signal length. We propose to use the perpendicular constraint to achieve this signal length-independent property. This constraint relates the form of the forward filters to that of the inverse filters. It is defined as,

$$\vec{c}\cdot\vec{r}\big|_{s=j} = \delta_j \qquad \vec{c}\cdot\vec{s}\big|_{s=j} = 0$$

$$\vec{d}\cdot\vec{r}\big|_{s=j} = 0 \qquad \vec{d}\cdot\vec{s}\big|_{s=j} = \delta_j \qquad (7)$$

where $\vec{a}\cdot\vec{b}$ denotes the vector product of $\vec{a}$ and $\vec{b}$, j is an even integer that ranges from 0 to $N-2$ and δ_k is the Dirac delta function which is equal to 1 when k=0 and 0 otherwise. As the signal length can be greater than or equal to the filter length, the length of the vectors defined in eqn.7 follows the signal length. To solve the length-dependency problem, eqn.7 should be true for all lengths greater than or equal to the filter length.

Eqn.7 is similar to the biorthogonal condition in the real field case. In particular, the forward lowpass filter is perpendicular to all the even shifts of the inverse bandpass filter, and the forward bandpass filter is perpendicular to all the even shifts of the inverse lowpass filter. The forward lowpass (bandpass) filter is also perpendicular to the even shift of the inverse lowpass (bandpass) filter, except for a zero shift. By imposing this constraint on binary filters, one could avoid the signal length-dependency problem.

Consider the case when the filter length is eight. The perpendicular constraint in eqn.7 is reduced to two sets of 14 equations (one for $\vec{r}$ and the other for $\vec{s}$),

$$\begin{bmatrix}
c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\
 & & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\
 & & & & c_0 & c_1 & c_2 & c_3 \\
 & & & & & & c_0 & c_1 \\
c_6 & c_7 & & & & & & \\
c_4 & c_5 & c_6 & c_7 & & & & \\
c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & & \\
d_0 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 \\
 & & d_0 & d_1 & d_2 & d_3 & d_4 & d_5 \\
 & & & & d_0 & d_1 & d_2 & d_3 \\
 & & & & & & d_0 & d_1 \\
d_6 & d_7 & & & & & & \\
d_4 & d_5 & d_6 & d_7 & & & & \\
d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & &
\end{bmatrix}
\begin{bmatrix}
r_0 & s_0 \\ r_1 & s_1 \\ r_2 & s_2 \\ r_3 & s_3 \\ r_4 & s_4 \\ r_5 & s_5 \\ r_6 & s_6 \\ r_7 & s_7
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0
\end{bmatrix} \qquad (8)$$

The empty entries in eqn.8 represent zeros. In general, the number of equations produced from the perpendicular constraint is (2M-2) where M is the filter length. Thus the size of the matrix produced would be (2M-2) x M. Due to eqn.4, this matrix would have a full rank for perfect reconstruction. However, it produces a set of over-determined equations for solving $\vec{r}$ and $\vec{s}$, and one needs to check for consistency for each solution obtained. Therefore, extra constraints on the binary filters should be imposed as part of the consistency checking procedure as shown in the next section. Note that the constraints resulting from the consistency checking procedure are different for different filter lengths [5].

4. FILTER DESIGN STRATEGY

Consider a filter design for a filter length of eight. There are 16 unknown coefficients. By eqn.3, the number of unknowns is reduced to nine, which gives 512 feasible designs. However, not all of them are length independent, as it can be seen from the choice in eqn.5. The perpendicular constraint is needed to determine those designs that are signal length independent.

According to eqn.3 and by manipulating the matrix in eqn.8, it can be shown that,

$$\begin{bmatrix}
0 & 1 \\
c_0 & c_1 \\
c_6 & c_7 \\
c_0 + c_2 & 1 + c_1 + c_3 \\
1 & 0 \\
d_0 & d_0 \\
d_6 & d_6 \\
1 + d_0 + d_2 & 1 + d_0 + d_2
\end{bmatrix}
\begin{bmatrix}
r_0 & s_0 \\ r_1 & s_1
\end{bmatrix}
=
\begin{bmatrix}
X_1 & Y_1 + Y_3 \\
X_2 & Y_2 \\
0 & 0 \\
X_3 & Y_3 \\
X_1 & Y_4 \\
0 & Y_5 \\
0 & 0 \\
0 & Y_6
\end{bmatrix} \qquad (9)$$

where

$$X_1 = c_0 + c_1$$

$$X_2 = (c_0 + c_1)(1 + c_6 + c_7)$$

$$X_3 = (c_0 + c_1 + c_2 + c_3)(c_4 + c_5)$$

$$Y_1 = V_2 + c_2(d_0 + d_2 + V_1 + V_2) + c_3(V_1 + V_2)$$

$$Y_3 = c_6(d_0 + d_2 + V_1 + V_2) + c_7(V_1 + V_2)$$

$$Y_4 = Y_1 + Y_3 + d_0$$

$$Y_5 = d_0(d_0 + d_2 + d_4)$$

$$Y_6 = d_4(d_0 + d_2)$$

$$V_1 = c_4 + (c_2 + c_4)(d_0 + d_4) + (c_0 + c_4)(c_2 + c_3 + c_4 + c_5)$$
$$V_2 = c_2 + (c_0 + c_2)(d_0 + d_4) + (c_0 + c_4)(c_0 + c_1 + c_2 + c_3)$$
$$V_3 = c_0(d_0 + d_4 + c_4) + c_1(c_0 + c_4)$$

This provides a set of over-determined equations for solving r_0, r_1, s_0, s_1. A check for consistency between these solutions is thus required. In particular, it can be shown that the consistency requires,

$$\begin{aligned}
&(c_0 + c_1)(c_2 + c_3 + c_4 + c_5) = 0\\
&(c_2 + c_3)(c_4 + c_5) = 0\\
&d_0(d_2 + d_4) = 0\\
&d_2 d_4 = 0\\
&c_0 d_2 + (c_0 + c_1)(Y_1 + Y_3) + Y_2 = 0\\
&c_6 d_2 + (c_6 + c_7)(Y_1 + Y_3) = 0\\
&c_2 d_2 + (c_2 + c_3)(Y_1 + Y_3) + Y_1 + Y_2 = 0
\end{aligned} \qquad (10)$$

Consider the choice in eqn.5, it does not satisfy the second and the fourth constraints. Therefore, the form of the inverse filter is length dependent. Solving eqn.10 gives the following 10 cases,

$$\begin{aligned}
&c_0 = c_1, c_2 = c_3, d_0 = 0, d_2 = 1, d_4 = 0, \{c_0 = 0 \text{ or } c_4 = 1 + c_5\}\\
&c_0 = c_1 = c_2 = c_3 = 0, d_0 = 0, d_2 = 0, d_4 = 0, \{c_4 = 0 \text{ or } c_5 = 0\}\\
&c_0 = c_1 = 0, c_2 = c_3, d_0 = 0, d_2 = 0, d_4 = 1, \{c_2 = 0 \text{ or } c_4 = 1 + c_5\}\\
&c_0 = c_1, c_2 = c_3, d_0 = 1, d_2 = d_4 = 0, \{c_4 = c_5 \text{ or } c_4 = c_0 + c_2\}\\
&c_0 = c_1, c_4 = c_5, d_0 = 0, d_2 = 1, d_4 = 0, c_2 = c_0 + c_4, c_3 = 1 + c_2\\
&c_0 = c_1, c_4 = c_5, d_0 = 0, d_2 = 0, c_2 = 1 + c_3, \{c_0 = 0 \text{ or } d_4 = 1\}\\
&c_0 = c_1, c_4 = c_5 = 0, d_0 = 1, d_2 = d_4 = 0, c_2 = 1 + c_3\\
&c_2 = c_3, c_4 = c_5 = 0, d_0 = 0, d_2 = 1, d_4 = 0, c_0 = c_2, c_1 = 1 + c_0\\
&c_2 = c_3, c_4 = c_5, d_0 = 0, d_2 = 0, c_1 = 1 + c_0, \{d_4 = 0 \text{ or } c_0 = c_2 + c_4\}\\
&c_2 = c_3 = c_4 = c_5 = 0, d_0 = 1, d_2 = d_4 = 0, c_1 = 1, c_0 = 0
\end{aligned} \qquad (11)$$

In summary, there are 66 feasible designs from eqn.11. They have the same number of vanishing moment, the same spectral behavior and are length independent. One could use the total number of "ones" in the filter coefficients to group them. There are four groups. They are summarized in Table 1. The expression for the inverse filters can be obtained by solving eqn.8 and eqn.9 using eqn.10. They are not shown here due to space limitation.

A close examination of eqn.11 reveals that the lowpass filters in a number of designs are the same while the bandpass filters are related by a two-circular shift. For example, in Group 1,

$$\begin{Bmatrix}\text{lowpass,}\\ \text{bandpass}\end{Bmatrix} = \begin{Bmatrix}
\text{a.} & \{0,1,0,0,0,0,0,0\}\{0,0,0,0,0,0,1,1\}\\
\text{b.} & \{0,1,0,0,0,0,0,0\}\{0,0,0,0,1,1,0,0\}\\
\text{c.} & \{0,1,0,0,0,0,0,0\}\{0,0,1,1,0,0,0,0\}\\
\text{d.} & \{0,1,0,0,0,0,0,0\}\{1,1,0,0,0,0,0,0\}
\end{Bmatrix} \qquad (12)$$

Their lowpass outputs are thus the same while their bandpass outputs are related by circular-shifts. Removing these shifted pairs, only 32 feasible designs remain as shown in Table 1. The four designs in Group 1 are,

$$\begin{Bmatrix}\text{lowpass,}\\ \text{bandpass}\end{Bmatrix} = \begin{Bmatrix}
\text{1.a} & \{0,0,0,0,0,0,0,1\}\{1,1,0,0,0,0,0,0\}\\
\text{1.b} & \{0,0,0,0,0,1,0,0\}\{1,1,0,0,0,0,0,0\}\\
\text{1.c} & \{0,0,0,1,0,0,0,0\}\{1,1,0,0,0,0,0,0\}\\
\text{1.d} & \{0,1,0,0,0,0,0,0\}\{1,1,0,0,0,0,0,0\}
\end{Bmatrix} \qquad (13)$$

The bandpass filters are the same while the lowpass filters are related by a circular shift. Therefore, their lowpass outputs at the first level of decomposition are related by one-circular shift. At the second level of decomposition, (1.a) and (1.c) take those samples with indexes $(4i+1)$ while (1.b) and (1.d) take those with indexes $(4i+3)$ for $i = 0,1,\cdots, N/4 - 1$. At the third level of decomposition, all lowpass outputs are different. Those samples taken by (1.a), (1.b), (1.c) and (1.d) are $(8i+1)$, $(8i+3)$, $(8i+5)$, $(8i+7)$ for $i = 0,1,\cdots, N/8 - 1$ respectively.

In principle, there is no difference between the even and odd number samples. Filters in the same group should share similar filter properties statistically. Among filters from different groups, one could use different criteria to choose a particular filter. If the computational speed is of the greatest concern, one could choose filters from Group 1.

5. SIMULATION RESULTS

To determine how it works, we applied the BWT to a set of binary images. Figure 1(a) shows one of these cases. The transformed result using a Group 2 filter is shown in Figure 1(b). Note that the transformed coefficients resulting from the BWT are binary. There is no expansion in their range. Most of the large value coefficients using the BWT correspond to high sequency transition. We can clearly see that the high sequency edge transitions in the character "h" were mapped into the high sequency regions in the transformed data. In particular, all three bandpass subbands show clearly the horizontal edges, the vertical edges and the diagonal edges of the original image.

In order to study the compactness of the signal representation produced by the BWT, the entropy is calculated. A small entropy indicates a large discrepancy between the number of zero and non-zero coefficients, and thus a potentially more efficient coding representation. Table 2 summarizes the entropy results when different binary filters are used. In general, the entropy was significantly reduced in the transformed images for all the binary filters, indicating a more compact image representation after the BWT.

6. IN-PLACE IMPLEMENTATION

The lifting implementation in the real-field wavelet transform allows the transform to be carried out efficiently in the spatial domain [6]. In particular, it enables an in-place implementation structure that not only reduces the computational complexity, but also saves the memory space involved in the transform. Similar to the case in the real field, the BWT has an in-place implementation structure in which the transform can be carried out efficiently in the binary field.

In order to have an in-place implementation structure, the odd and the even samples of the original signal are split into two sequences. These two sequences are then updated according to the binary filter coefficients. This is similar to the "split, update and predict" procedure of the lifting implementation in the real field.

As discussed above, the lowpass filter in Group 1 involves a sub-sampling operation only, while the bandpass filter involves an exclusive-or operation between two neighboring samples. The in-place implementation structure is thus trivial for the Group 1 filter. The in-place implementation structure can also be extended to filters involving not only sub-sampling operations. Let us consider filters from Group 3 (cf. Table 1 and Figure 2). The in-place implementation consists of three stages. In the first stage, the operation is similar to that in Group 1 filters. The result is then circular shifted by one. Two more exclusive-or operations are then applied to the shifted sequence, which gives the desired filter outputs. The total number of exclusive-or operation is reduced from five to three using this in-place structure. Filters in Groups 2 and 4 can be implemented using the same philosophy.

	No of XOR	No of feasible designs	No of feasible designs (excluding circular shifted pairs)	Filter example (lowpass, bandpass)
Group 1	1	16	4	{0,1,0,0,0,0,0,0} {1,1,0,0,0,0,0,0}
Group2	3	28	12	{1,1,1,0,0,0,0,0} {1,1,0,0,0,0,0,0}
Group 3	5	18	12	{1,1,1,1,0,0,0,1} {1,1,0,0,0,0,0,0}
Group4	7	4	4	{1,1,1,1,1,1,1,0} {1,1,0,0,0,0,0,0}

Table 1: Filter Grouping for filter length equals to eight.

Image	Original Entropy	Filter Length = 8			
		Group 1	Group 2	Group 3	Group 4
"a"	0.8946	0.1629	0.2456	0.3079	0.3580
"b"	0.9079	0.0935	0.1426	0.1811	0.2181
"c"	0.8809	0.0624	0.0806	0.0963	0.1136
"d"	0.7567	0.0653	0.0980	0.1079	0.1342
"e"	0.7279	0.1008	0.1594	0.2101	0.2489
"g"	0.9548	0.1044	0.1600	0.2051	0.2394
"h"	0.7425	0.1172	0.1898	0.2446	0.2842
"i"	0.7645	0.0909	0.1452	0.1899	0.2215
"j"	0.4780	0.0767	0.1318	0.1763	0.2163
"m"	0.8941	0.0956	0.1343	0.1624	0.1907
"n"	0.8638	0.0832	0.1359	0.1765	0.2154
"o"	0.8108	0.1113	0.1795	0.2367	0.2788
"s"	0.3661	0.2299	0.2177	0.2382	0.2317

Table 2: Summary of entropy values for the original image and the transformed images using filters from different groups.

7. CONCLUSIONS

In order to maintain an invertible binary wavelet transform with the desirable multiple resolution properties, three constraints are required, namely the bandwidth, the perfect reconstruction and the vanishing moment constraints. While these constraints guarantee that the transform is invertible, the transform is constrained to being non-orthogonal and non-biorthogonal. This renders the inverse filter design non-trivial and the form of the inverse filters could be signal length dependent.

This paper proposes to use the perpendicular constraint to resolve the problem. In particular, this constraint requires that the forward lowpass (bandpass) filters be perpendicular to all the even shifts of the inverse bandpass (lowpass) filters. The forward lowpass (bandpass) filters are also required to be perpendicular to all the even shifts of the inverse lowpass (bandpass) fitlers except for the zero shift. With this constraint in the filter design, a set of conditions for length independency is derived. An efficient implementation structure of the binary filters has also been explored. This implementation has the in-place property, which saves the memory space involved in the transform and reduces the computational complexity.

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(a) (b)

Figure 1: (a) The binary image and (b) the transformed image using (b) Group 2 filter with a three level of decomposition.

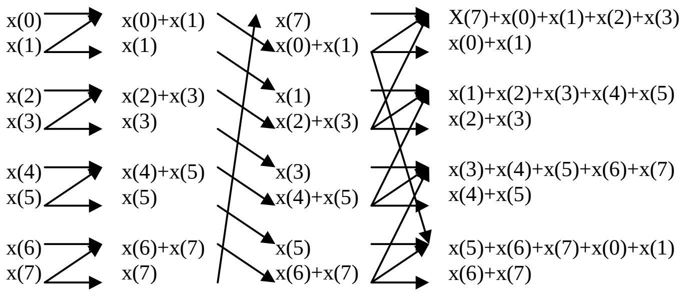

Figure 2: An in-place implementation for Group 3 filter. The sign "+" means a modulo-2 addition which is equivalent to an exclusive-or operation.

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