OPTIMAL PREFILTERS FOR THE MULTIWAVELET FILTER BANKS

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ABSTRACT

This paper proposes a method to obtain optimal 2^{nd}-order approximation preserving prefilters for a given orthogonal unbalanced multiwavelet basis. This procedure uses the prefilter construction introduced in [3]. The prefilter optimization scheme exploits the Taylor series expansion of the prefilter combined with the multiwavelet. Using the DGHM multiwavelet with the obtained optimal prefilter, we find that quadratic input signals are annihilated by the high-pass portion of filter bank at the first level of decomposition.

1. INTRODUCTION

One of the most important properties of multiwavelet is its approximation order. In the case of compactly supported multiwavelets, this corresponds to the property of polynomial reproduction. Since the multiwavelets have more than one scaling functions, the dilation equation becomes the dilation equation with matrix coefficients. Thus, in applications, one must associate a given discrete signal into a sequence of length-r vectors (where r is the number of scaling functions) without losing some certain properties of the underlying multiwavelet. Such a process is referred to as *prefiltering* or *multiwavelet initialization*. One prefiltering method for the DGHM multiwavelet suggested by Geronimo is to create a function with vector sequence of length r based on the interpolating property of the DGHM scaling functions. It yields a prefilter which is approximation order preserving but not orthogonal. In [3], Hardin and Roach develop a theory for constructing prefilters which preserve both orthogonality and approximation order up to order 2. It has been shown in [1, 3, 7] that choosing a prefilter is a crucial step which significantly affects the performance of the multiwavelet filter bank. In this paper, we use the results in [3] to construct the orthogonal length-3 approximation order preserving prefilter. Since an infinite number of such prefilters can be constructed, we propose a criterion to find the optimal prefilter for a given orthogonal multiwavelet basis. The criterion exploits the Taylor series expansion of the prefilter combined with the multiwavelet.

2. MULTIWAVELET PRELIMINARIES

Let Φ denote a compactly supported orthonormal scaling vector

$$\Phi = (\phi^1,\ \phi^2, \ldots, \phi^r)^T$$

[†] This work was supported in part by a grant from the National Science Foundation.

where r is the number of scalar scaling functions. Then $\Phi(t)$ satisfies a two-scale dilation equation of the form

$$\Phi(t) = \sqrt{2}\sum_n h(n)\Phi(2t-n) \tag{1}$$

for some finite sequence h of $r \times r$ matrices. Furthermore, the integer shifts of the components of Φ form an orthonormal system, that is

$$< \phi^l(\cdot - n), \phi^{l'}(\cdot - n') >= \delta_{l,l'}\delta_{n,n'}. \tag{2}$$

Let V_0 denote the closed span of $\{\phi^l(\cdot - n) \mid n \in \mathbf{Z}, l = 1, 2, \ldots, r\}$ and define $V_j = \{f(\frac{\cdot}{2^j}) \mid f \in V_0\}$. Then $(V_j)_{j\in\mathbf{Z}}$ is a multiresolution analysis of $L^2(\mathbf{R})$ [5]. Note we choose the decreasing convention $V_{j+1} \subset V_j$.

Let W_j denote the orthogonal complement of V_j in V_{j-1}. Then there exists an orthogonal multiwavelet Ψ=$(\psi^1,\ \psi^2, \ldots, \psi^r)^T$ such that $\{\psi^l(\cdot - n) \mid l = 1, 2, \ldots, r$ and $n \in \mathbf{Z}\}$ forms an orthonormal basis of W_0. Since $W_0 \subset V_{-1}$, there exists a sequence g of $r \times r$ matrices such that

$$\Psi(t) = \sqrt{2}\sum_n g(n)\Phi(2t-n). \tag{3}$$

Let $f \in V_0$, then f can be written as a linear combination of the basis in V_0.

$$f(t) = \sum_{k\in\mathbf{Z}} \mathbf{c_0}(k)^T\Phi(t-k) \tag{4}$$

for some sequence $\mathbf{c_0} \in l_2(\mathbf{Z})^r$. Since $V_0 = V_1 \oplus W_1$, f can also be expressed as

$$f(t) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbf{Z}} \mathbf{c_1}(k)^T\Phi(\frac{t}{2}-k) + \frac{1}{\sqrt{2}}\sum_{k\in\mathbf{Z}} \mathbf{d_1}(k)^T\Psi(\frac{t}{2}-k). \tag{5}$$

The coefficients $\mathbf{c_1}$ and $\mathbf{d_1}$ are related to $\mathbf{c_0}$ via the following decomposition and reconstruction algorithm:

$$\mathbf{c_1}(k) = \sum_n h(n)\mathbf{c_0}(2k+n) \tag{6}$$

$$\mathbf{d_1}(k) = \sum_n g(n)\mathbf{c_0}(2k+n) \tag{7}$$

$$\mathbf{c_0}(k) = \sum_n h(k-2n)^T\mathbf{c_1}(n) + \sum_n g(k-2n)^T\mathbf{d_1}(n). \tag{8}$$

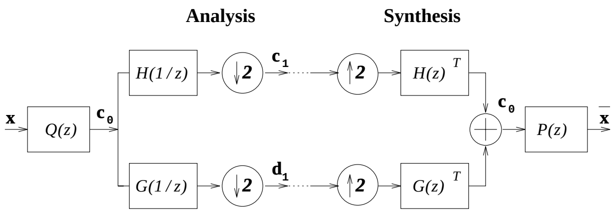


Fig. 1. Multiwavelet filter bank.

Let $Q(z)$ be the z transform of the matrix-valued sequence q. If $Q(z)$ can be written in the form

$$Q(z) = \sum_{n=l}^{m} q_n z^{-n} = q_l z^{-l} + ... + q_m z^{-m}$$

where q_l and $q_m \neq 0$, then $Q(z)$ is said to have a length of $(m - l + 1)$.

3. MULTIWAVELET PREFILTERS AND OPTIMIZATION CRITERION

The block diagram of a multiwavelet filter bank can be shown as in Figure 1 where $Q(z)$ and $P(z)$ represent the prefilter and the postfilter, respectively. Vector sequence $\mathbf{x}$ is obtained by the following operator. Define the operator $D_r : \mathbf{R}^{\mathbf{Z}} \to (\mathbf{R}^{\mathbf{r}})^{\mathbf{Z}}$ which partitions a scalar sequence into a sequence grouped in vectors of length r as follows. Given a scalar sequence $x(n),\ n \in \mathbf{Z}$, then $\mathbf{x} = D_r(x)$ is given by

$$\begin{aligned} \mathbf{x} = D_r(x) &= (\downarrow r) \begin{pmatrix} x(n) \\ x(n+1) \\ \vdots \\ x(n+r-1) \end{pmatrix}_{n\in\mathbf{Z}} \\ &= \begin{pmatrix} x(rn) \\ x(rn+1) \\ \vdots \\ x(rn+r-1) \end{pmatrix}_{n\in\mathbf{Z}} . \end{aligned}$$

The block diagram of the high-pass portion of the analysis multiwavelet filter bank is shown in Figure 2a. By using the first Nobel Identity, the block diagram in Figure 2a is equivalent to the one shown in Figure 2b. Let

$$W(z) = \begin{pmatrix} W_1(z) \\ \vdots \\ W_r(z) \end{pmatrix} = G(1/z^r)Q(z^r) \begin{pmatrix} z^0 \\ \vdots \\ z^{r-1} \end{pmatrix} . \quad (9)$$

Then from Figure 2b, we see that $V(z) = W(z)X(z)$. The *energy compaction ratio* is defined as the ratio of the total energy of the output from the high-pass portion of the analysis filter bank and the total energy of input signal. Then, if $X(z)$ is stationary, $V(z)$ is stationary as well. So the total energy of $V(z)$ is $2r$ times the energy of $U(z)$. Thus the energy compaction ratio is obtained by

$$\text{Energy compaction ratio} = \frac{\int |X(e^{j\omega})|^2 (|W_1(e^{j\omega})|^2 + \ldots + |W_r(e^{j\omega})|^2)\, d\omega}{2r \int |X(e^{j\omega})|^2 d\omega}.$$

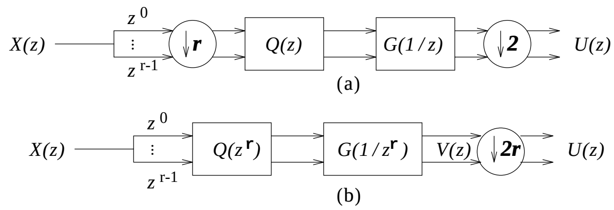


Fig. 2. (a) Block diagram of high-pass portion of analysis filter bank. (b) Equivalent system.

The energy compaction ratio can be used to see how effectively the high-pass portion of the orthogonal filter bank annihilates the input signal. Thus, the smaller the ratio, the better the energy compaction.

It is known [4] that if Φ is compactly supported, Φ has approximation order p if and only if there exist vector coefficients $\alpha_n(k)$ such that

$$t^n = \sum_k \alpha_n(k)^T \Phi(t-k)\ , n = 0, ..., p-1 \quad (10)$$

where $\alpha_n = (\alpha_n^1\ \alpha_n^2\ ...\ \alpha_n^r)^T$. Furthermore, it follows from (10) that the high-pass filter g annihilates α_n for $n = 0, 1, ..., p-1$, i.e.,

$$\breve{g} * \alpha_n = \mathbf{0},\ n = 0, 1, ...p-1 \quad (11)$$

where $\breve{g}(k) = g(-k)$.

Let $S : C(\mathbf{R}) \to \mathbf{R}^{\mathbf{Z}}$ be the sampling operator $S(f) = (f(\frac{n}{r}))_{n\in\mathbf{Z}}$ and let $\Lambda : C(\mathbf{R}) \to (\mathbf{R}^{\mathbf{r}})^{\mathbf{Z}}$ be defined by $\Lambda(f) = \frac{1}{\sqrt{r}} D_r(S(f))$. Let $\pi_n(t) := t^n$ and $a_n := \Lambda(\pi_n)$.

A prefilter $Q(z)$ with impulse response q is said to be a p^{th}-*order approximation preserving prefilter for* Φ if [3]

$$q * a_n = \alpha_n \text{ mod span}\{\alpha_0, \ldots, \alpha_{n-1}\},\ n = 0, \ldots, p-1. \quad (12)$$

Note: $f = g$ mod L if and only if $f - g \in L$.

Lemma 1 *Suppose Φ has approximation order p and q is a p^{th}-order approximation preserving prefilter for Φ. Let $W(z)$ be given by (9). Then, $W^{(n)}(1) = \mathbf{0}$ for $n = 0, ..., p-1$. (Here $W^{(n)}$ denotes the n^{th} derivative of W.)*

Proof : Let w be the inverse Z-transform of $W(z)$ and let $p_n = S(\pi_n)$. Then, by the first Nobel Indentity, $(\downarrow r)(w * p_n) = \breve{g} * q * D_r(p_n)$. Since $a_n = \Lambda(\pi_n) = D_r(p_n)$ and $\alpha_n = q * a_n$, we have $(\downarrow r)(w*p_n) = \breve{g} * \alpha_n$ and, hence, by (11), $(\downarrow r)(w*p_n) = \mathbf{0}$ for $n = 0, ..., p-1$. The shift invariance of π_n then implies $w * p_n = \mathbf{0}$ for $n = 0, ..., p-1$.

Therefore, $(w * p_n)(0) = \sum_k w(k)(0-k)^n = (-1)^n \sum_k w(k)k^n = \mathbf{0}$ for $n = 0, ..., p-1$. Thus,

$$\sum_k w(k)(a_0 + a_1 k + ... + a_{p-1}k^{p-1}) = \mathbf{0} \quad (13)$$

for any $a_i \in \mathbf{R}$. Since $W(z) = \sum_k w(k)z^{-k}$, we have

$$W^{(n)}(1) = \sum_k w(k) \quad \text{for } n = 0 \quad (14)$$

Table 1. Optimal prefilter coefficients for the DGHM multiwavelet

a	q(1)	0.312146768057	-0.111458514406
		-0.526851707882	0.188123391564
	q(0)	0.485328400031	-0.584495172202
		0.584495172202	0.485328400031
	q(-1)	0.188123391564	0.526851707882
		0.111458514406	0.312146768057
b	q(1)	0.009390110250	-0.067942050642
		0.002264535743	-0.016385026171
	q(0)	0.992593475574	-0.098895392339
		0.098895392339	0.992593475574
	q(-1)	-0.016385026171	-0.002264535743
		0.067942050642	0.009390110250
c	q(1)	-0.626283326739	0.299036923482
		0.632561984796	-0.302034848716
	q(0)	0.155146499297	0.053999651374
		0.053999651374	-0.155146499297
	q(-1)	0.302034848716	0.632561984796
		0.299036923482	0.626283326739
d	q(1)	0.247052825476	0.164291169222
		0.789564730613	0.525063870525
	q(0)	0.108909066323	0.031742659818
		0.031742659818	-0.108909066323
	q(-1)	-0.525063870525	0.789564730613
		0.164291169222	-0.247052825476

For $n > 0$,

$$W^{(n)}(1) = \sum_k w(k)(-k)(-k-1)...(-k-(n-1)) \quad (15)$$

Thus, from (13), (14) and (15), it is clear that

$$W^{(n)}(1) = \mathbf{0} \quad \text{for } n = 0, ..., p-1.$$

Since the spectra of most natural signals are concentrated around zero frequency, a natural way to obtain a small energy compaction ratio is to find a prefilter such that $W(e^{j\omega})^* W(e^{j\omega})$ is zero and as flat at zero frequency as possible. Now consider the Taylor series expansion of $W(e^{j\omega})$ about $\omega = 0$:

$$W(e^{j\omega}) = \sum_{n=0}^{\infty} c_n \omega^n \quad (16)$$

where c_n is the $r \times 1$ vector given by $c_n(i) = \frac{W_i^{(n)}(1)}{n!}$, $i = 1, ..., r$. For a prefilter q, we let $m = m(q)$ denote the index of the first nonzero coefficient c_m in (16). If Φ has approximation order p and q is p^{th}-order approximation preserving, then, by Lemma 1, $c_0 = c_1 = ... = c_{p-1} = \mathbf{0}$ and so we have $m(q) \geq p$. Observe that

$$W(e^{j\omega})^* W(e^{j\omega}) = c_m^T c_m \, \omega^{2m} + \mathcal{O}(\omega^{2m+1})$$

which leads us to consider the following:
Optimization Criterion. Given a collection L of prefilters, let

$$m_L := \max_{q \in L} m(q)$$

be the largest possible m for any of the prefilters in L. If q is a prefilter with $m(q) = m_L$ that also minimizes $\|c_{m_L}\|^2 = c_{m_L}^* c_{m_L}$ then we say that q is optimal (with respect to L).

Table 2. Taylor series coefficients $c_0(1) - c_4(1)$ of $W(e^{j\omega})$ using the optimal prefilters in Table 1.

	$c_0(1)$	$c_1(1)$	$c_2(1)$	$c_3(1)$	$c_4(1)$
a	0	0	0	-j0.40825	-1.17113
b	0	0	0	-j0.40825	0.01706
c	0	0	0	-j0.40825	0.37899
d	0	0	0	-j0.40825	3.81684

Table 3. Taylor series coefficients $c_0(2) - c_4(2)$ of $W(e^{j\omega})$ using the optimal prefilters in Table 1.

	$c_0(2)$	$c_1(2)$	$c_2(2)$	$c_3(2)$	$c_4(2)$
a	0	0	0	-j0.92446	0.56882
b	0	0	0	-j0.25148	0.44090
c	0	0	0	j0.69748	-2.08410
d	0	0	0	j2.61059	-3.52904

We next apply the optimization criterion to find the optimal length-3 approximation order preserving prefilter for the DGHM multiwavelet. For the DGHM multiwavelet ($r = 2$), $W_1(e^{j\omega})$ and $W_2(e^{j\omega})$ are the Fourier transform of the prefilter combined with the antisymmetric and symmetric wavelets, respectively. Since the multiwavelet has approximation order 2, c_0 and c_1 are zero vectors. Note that, from observation, $c_2(1)$ is automatically zero for the DGHM multiwavelet. Following the optimization criterion, we then search for a prefilter that minimizes $|c_2(2)|^2$. Using the prefilter construction given in [3], we found several prefilters such that $c_2(2)$ was zero as well. This implies that quadratic input signals are annihilated by $W(e^{j\omega})$. Table 1 gives the optimal prefilter coefficients. Tables 2 and 3 show the Taylor series coefficients $c_0(1) - c_4(1)$ and $c_0(2) - c_4(2)$ using the obtained prefilters, respectively.

4. RESULT WITH IMAGE COMPRESSION ALGORITHM

In this section, we apply the multiwavelet filter bank to the image compression algorithm using the obtained optimal length-3 approximation order preserving prefilters. The image compression scheme used in this paper is an adaptation of the binary-uncoded SPIHT algorithm of [6] which exploits the zero-tree structure of wavelet coefficients. The results were obtained with gray-scaled, 8 bpp, 512x512 Lena image. We first obtain the results of the image compression using DGHM multiwavelet with the optimal prefilters. Table 4 shows the PSNR of the decompressed Lena image at different bit rates. From the image compression results and the Taylor series coefficients of the optimal prefilters, it can be seen that the best prefilter amoung the obtained optimal prefilters (prefilter b) is the one that has small nonzero Taylor series coeffcients at the low order of ω. In [3], six possible length-3 quasi-interpolation prefilters and four possible length-2 approximation order preserving prefilters for the DGHM multiwavelet are given. We next compare the result with the optimal length-3 quasi-interpolation prefilter and the optimal length-2 approximation order preserving prefilter. Additionally, comparisons are made with the Daubechies-4 scalar wavelet which has the same approximation order ($p = 2$). Table 5 shows the PSNR of the decompressed Lena image using the DGHM multiwavelet with various prefilters and the D-4 scalar wavelet. Figure 3 shows the frequency

Table 4. PSNR comparison of decompressed Lena image using optimal prefilters.

CPR	PNSR (dB)			
	Optimal prefilters			
	a	b	c	d
8:1	38.88	39.21	38.59	37.46
16:1	35.31	35.66	35.02	33.62
32:1	31.93	32.12	31.67	30.58
64:1	29.10	29.11	28.99	28.41
128:1	26.64	26.53	26.62	26.37

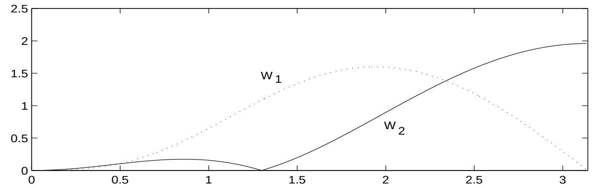
Table 5. PSNR comparison of the decompressed Lena images using the DGHM multiwavelet and the Daubechies-4.

CPR	PNSR (dB)			
	DGHM			D-4
	Opt l-3 quasi	Opt l-2 approx	Opt l-3 approx	
8:1	38.85	38.85	39.21	38.74
16:1	35.25	35.30	35.66	35.19
32:1	31.88	31.98	32.12	31.85
64:1	29.00	29.17	29.11	29.02
128:1	26.54	26.64	26.53	26.51

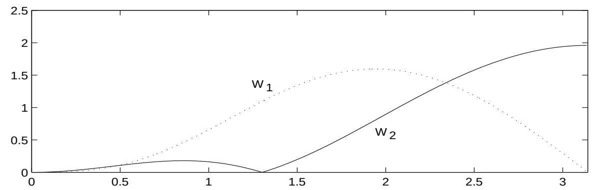
responses of $W_1(e^{j\omega})$ and $W_2(e^{j\omega})$ when $Q(z)$ is the optimal length-3 approximation order preserving prefilter (prefilter b) and compares with the responses when $Q(z)$ are other prefilters. The result of image comression agrees with the frequency responses in Figures 3 which show that the frequency response of the optimal length-3 approximation order preserving prefilter is the flattest at low frequency.

5. CONCLUSIONS

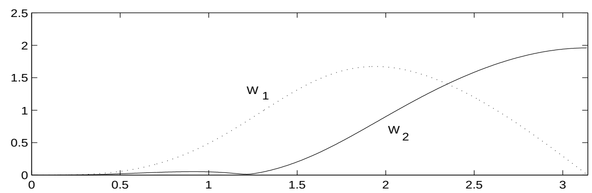
This paper has developed an optimization criterion to find the optimal 2^{nd}-order approximation preserving prefilter for a given orthogonal multiwavelet basis based on the taylor series expansion of the prefilter combined with the multiwavelet. The results show that the DGHM multiwavelet with the obtained optimal prefilter outperforms other prefilters which were included in this study and the D-4 scalar wavelet with the same approximation order.



(a) Optimal length-3 quasi-interpolation prefilter



(b) Optimal length-2 approximation order preserving prefilter



(c) Optimal length-3 approximation order preserving prefilter

Fig. 3. Frequency responses of the $W_1(e^{j\omega})$ and $W_2(e^{j\omega})$ of the DGHM multiwavelet with prefilters

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