# ACTIVITY DETECTION IN UNKNOWN NOISE ENVIRONMENT

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**ABSTRACT**

In many applications there exists an array of cells (or bins), each containing either an activity (signal) plus noise, or noise only. A common problem is to identify the active bins, assuming that the noise level in the array is unknown. In this paper we present a novel approach for solving this problem. The approach is based on two steps. In the first, we estimate the noise level and in the second we perform a sequential test to decide, for each bin, whether it is active or not. We show that the proposed algorithm collapses to well known special cases. The performance of the proposed algorithm is analyzed analytically and is demonstrated via simulation results.

## 1. INTRODUCTION AND PROBLEM FORMULATION

Consider the additive noise model:

$$x_i = s_i + n_i \qquad i = 1, \ldots, N \tag{1}$$

where $\{s_i\}_{i=1}^N$ are unknown constants. Assume, also that $n_i$ is a real, zero mean, stationary, white Gaussian random process with unknown variance, denoted by $\sigma^2$. We refer to measurements with $s_i = 0$ as noise only measurements, while the other measurements are refereed to as activity measurements. The aim is to identify the active bins, or the bin indices of the noise only measurements, "as good as possible". The total number of active bins is unknown. However, it is assumed that this number is bounded by $M < N$, where M is known a-priori.

The problem relates to many different applications. For example, in image denoising [6], $x_i$ is the $i$-th coefficient of the wavelet transform of the received signal (image), and $s_i$ is the $i$-th coefficient of the wavelet transform of the original signal (image). $n_i$ is the additive Gaussian noise with unknown variance, *i.e.* unknown noise level. Another application is synchronous $CDMA$ communication, where the measurements, $x_1, \ldots, x_N$, represent the outputs of the time shifted $PN$ sequence, matched filtered with the received signal. It is well known that $x_1, \ldots, x_N$ are independent, Gaussian random variable with mean $s_i \geq 0$ and common variance $\sigma^2$ [8]. Each index $i$ such that $s_i > 0$ corresponds to one replica of the signal present at that time shift.

Denote by $I$ the set of indices such that $s_i = 0$, that is, $I = \{i | s_i = 0\}$. Recall that the unknown size of $I$ is at least N-M. As stated earlier, the aim is to detect/identifiy $I$ "as good as possible". In image denoising, detecting this set enables one to "clean" the image by performing the inverse wavelet transform without the coefficients belonging to $I$, since these coefficients do not contain any information about the original signal (image). In $CDMA$ communications, by detecting $I$ one knows which time shifts contain replica of the signals. These time shifts can be coherently combined to increase the signal to noise ratio of the received signal.

The Neyman-Pearson criterion is the most widely used criterion to assess the optimailty of a detector. However, since the problem of interests involves a series of detection problems, no equivalent criterion exists. Thus, optimality needs to be defined when trying to identify $I$ "as good as possible" based on $x_1, \ldots, x_N$. Denote by $\hat{I}$ the estimated set of indices belonging to $I$. Denote by $I - \hat{I}$ the set of indices which belongs to $I$ but do not belong to $\hat{I}$. We refer to the indices which belongs to $I - \hat{I}$ as "false alarms", *i.e.*, noise measurements which were detected as activity measurements. In the applications we refer to, one does not have any preferences on the exact indices belonging to $I - \hat{I}$. Therefore, we seek for an algorithm which is invariant to the exact location of the false bins. That is, a noise measurement in a specific bin should have an equal probability to be detected as an activity, as another noise measurement in a different bin. This reasoning leads to an optimality criterion similar to the Neyman-Pearson criterion per bin: *maximize the probability of detection of an activity, subject to a constant probability of false alarm which is the same for each index belonging to $I$, independently*.

In the sequel, we present a novel approach for solving the problem of interest. First, we estimate the noise level, $\sigma^2$. Then we perform a sequential binary hypothesis test to decide, for each bin, whether it is active or not.

The paper is organized as follows: In section 2, we discuss the difficulties with a different, well known approach to solve our problem - the MDL approach. In section 3, we describe the proposed noise level estimator (step 1), and in section 4 we present the entire algorithm (step 2). Section 5 provides simulation results and in section 6 we discuss the results and their relation to other, well know problems and solutions.

## 2. THE MDL ESTIMATOR

Our problem can be viewed as one of choosing the best model out of several possible models. The MDL [10] is the most common approach to deal with such problems. The MDL is an information theoretic criterion which chooses the model that minimizes the description length of both the data and the model. In our problem, the MDL estimate is the one which minimizes the following metric:

$$\begin{aligned} MDL(i) &= \min_{i=0,\ldots,N} -\log f_X(x_1, \ldots, x_N | \underline{\hat{\theta}}_i) + \\ &+ \quad 0.5 \cdot (2i+1) \log N \end{aligned} \tag{2}$$

where $\underline{\hat{\theta}}_i$ is the ML estimate of the unknown parameter vector assuming the existence of $i$ signals (activities). The unknown parameters are the $i$ locations of the activities, their $i$ corresponding levels, and the noise level.

Denote by $x_{1:N} \leq \cdots \leq x_{N:N}$ the *ordered* sample, and by $|x|_{1:N} \leq \cdots \leq |x|_{N:N}$ the *ordered* absolute values of the measurements. It can be shown that the ML estimates for the signal levels are the values of the $i$ largest absolute value measurements. The ML estimates for the locations of the signals are their corresponding indices. The ML estimate for the noise level is given by $\frac{1}{N}\sum_{i=1}^{N-i}|x|^2_{N-i+1:N}$. Clearly, the mean of the ML estimate of the noise level is inherently biased, unless $i = 0$. It can be proved this bias causes large probability of false alarm.

## 3. CENSORING BASED NOISE LEVEL ESTIMATION

In this section we give a brief description of an approximated order statistics maximum likelihood method for estimating the noise level from censored samples. This method was first described in [1]. In [3] we describe few other methods for accomplishing this task.

Assume that $x_1, \ldots, x_N$ are samples of an $i.i.d.$ zero mean, Gaussian random variables with variance $\sigma^2$ and that $x_{1:N} \leq x_{2:N} \leq \cdots \leq x_{N-1:N} \leq x_{N:N}$ are the ordered sample. In censoring based estimation one estimates the unknown parameters from a censored sample. Assume that we censor the smallest $r$ samples and the largest $s$ samples, and we aim to estimate $\sigma^2$ based on $x_{r+1:N}, x_{r+2:N}, \ldots, x_{N-s:N}$ only.

The likelihood of $x_{r+1:N}, x_{r+2:N}, \ldots, x_{N-s:N}$ is given by:

$$L(\sigma) = \frac{N!}{r!s!}\sigma^{-N+r+s}\left[F_X\left(\frac{x_{r+1:N}}{\sigma}\right)\right]^r \left[1-F_X\left(\frac{x_{N-s:N}}{\sigma}\right)\right]^s \prod_{i=r+1}^{n-s} f_X\left(\frac{x_{i:N}}{\sigma}\right) \tag{3}$$

The $ML$ estimate for the noise level is given by the solution of:

$$\frac{\partial \log L(\sigma)}{\partial \sigma} = -\frac{1}{\sigma}\left[A + \frac{r z_{r+1:N} f_X(z_{r+1:N})}{F_X(z_{r+1:N})} - \right. \tag{4}$$
$$\left. -\frac{s z_{N-s:N} f_X(z_{N-s:N})}{1-F_X(z_{N-s:N})} - \sum_{i=r+1}^{N-s} \frac{z_{i:N} f'_X(z_{i:N})}{f_X(z_{i:N})}\right] = 0$$

where $z_{i:N} = \frac{x_{i:N}}{\sigma}$, and $A = N - r - s$. Since no closed form expression for the distribution function of normal random variable exists, (**??**) can only be solved numerically.

In [1] it was suggested to expand the functions $\frac{f(z_{r+1:N})}{F(z_{r+1:N})}$ and $\frac{f(z_{N-s:N})}{1-F(z_{N-s:N})}$ about the points $\xi_{r+1} = F^{-1}(p_{r+1})$ and $\xi_{N-s} = F^{-1}(p_{N-s})$, where $p_{i+1} = \frac{i}{N+1}$. Expanding (5) using a Taylor series and plugging in the well known fact that for standard normal random variable $f'(z) = -zf(z)$, result in an approximation for (5). The solution of this equation is the approximated ML estimate of the noise level:

$$\hat{\sigma} = \frac{-D + \sqrt{D^2 + 4AE}}{2A} \tag{5}$$

where

$$D = r\alpha x_{r+1:N} - s\gamma x_{N-s:N} \tag{6}$$

$$E = \sum_{i=r+1}^{N-s} x^2_{i:N} + r\beta x^2_{r+1:N} + s\delta x^2_{N-s:N} \tag{7}$$

where $\alpha,\ \beta,\ \gamma$, and $\delta$ are given in the Appendix.

## 4. THE ALGORITHM AND ITS PERFORMANCE

In this section we present two variations of the proposed algorithm. The first algorithm is an adaptation of the $OS-CFAR$ detector [7] for our problem. The second algorithm presents a novel iterative approach which overcomes the disadvantage of the first one.

### 4.1. Algorithm I

Recall that the algorithm aims to detect the activity measurements as good as possible. That is, to identify the indices belonging to $I$ with a desired probability of false alarm. The proposed algorithm consists three stages:

1. Estimation.
2. Setting a threshold.
3. Decision.

*Estimation:* In this stage the noise level is estimated using (5). First, some of the smallest and largest samples are censored. The censoring must insure that, with high probability, the resulting measurements consist of noise only measurements. Let $x_{r+1:N}, \ldots, x_{N-s:N}$ be the censored sample. Under the assumption that all these measurements belong to the noise subspace (with high probability ), it is easy to verify that [2]:

$$f(x_{r+1:N}, \ldots, x_{n-s:N}|\sigma^2, I, \mu_1, \ldots, \mu_{|I|}) \approx f(x_{\tilde{r}+1:\tilde{N}}, \ldots, x_{\tilde{N}-\tilde{s}:\tilde{N}}|\sigma^2) \tag{8}$$

where $\tilde{N}$ is the actual number of noise only measurements and $\tilde{r}, \tilde{s}$ are the actual number of smallest and largest noise only censored samples, respectively. *For example,* assume that $N = 10$ and that $s_1 = 5, s_2 = -5, s_i = 0, i > 2$. In this example we have one negative and one positive activity (signal). The size of the noise sample is 8, thus $\tilde{N} = 8$. Also, if we censor the three highest and the three lowest measurements ($r = s = 3$), then with high probability, the two lowest measurements and the two highest measurements from the noise measurements are censored, so $\tilde{r} = \tilde{s} = 2$. The approximation in (8) enables us to use (5) for estimating the noise level.

It turns out that the false alarm rate of the algorithm is highly dependent on $\tilde{N}, \tilde{r}, \tilde{s}$. As an example, consider the case where $N = 100$, $s_1 > 0$ and $s_i = 0, \forall i > 1$. Assume that we have censored from the measurements the $25$ highest and lowest samples. The samples used to estimate the noise level are therefore $x_{26:99}, \ldots, x_{75:99}$. If one wrongly assumes that there are two activities, one positive and one negative he/she will assume that samples $x_{25:98}, \ldots, x_{75:98}$ are used for the noise level estimation. The resulting noise level estimate will be biased, which, in turn, results in a different (yet constant) probability of false alarm.

*Setting the threshold:* In this stage a threshold, denoted by $T$, is set. The threshold is of the form $T = a\hat{\sigma}^2$, where $a$ is set to insure a desired probability of false alarm.

*Decision:* In the decision stage each measurement, $x_i$, is compared to the threshold $T$. We distinguish between two cases. In the first case it is known that the activities are only positive (e.g., power). The decision rule is then of the form: "if $x_i < T$ then $i \in I$; otherwise $i \in \bar{I}$". In the second case activities can be positive or negative so we use the following decision rule: "if $|x_i| < T$ then $i \in I$; otherwise $i \in \bar{I}$".

### 4.2. Algorithm II

The main drawback of the proposed algorithm is the necessity to know the number of positive and negative activities to set a threshold for a desired false alarm rate. This unrealistic requirement prevents the ability to design an algorithm with a known false alarm probability, (although the algorithm is still of a constant false alarm rate). To overcome this difficulty we present a iterative version of the proposed algorithm.

The iterative algorithm uses the estimated number of sources obtained in one iteration as the input for the noise level estimator in the next iteration. As will be explained later, we can reduce the bias in noise level estimation in each iteration and thus to achieve the designed performance.

The new algorithm can be described in the following general scheme:

1. Estimation.
2. Setting a threshold.
3. Decision.
4. Stopping.
5. Return to 1.

*Estimation:* This stage is similar to the estimation stage in algorithm I. However, now it is repeated several times, each time with the estimated $\tilde{r}, \tilde{s}, \tilde{N}$ from the previous iteration.

*Setting a threshold and decision:* These stages are the same as in the previous algorithm.

*Stopping:* In this stage we decide whether to make another iteration or not. We suggest the following, simple criterion: if no new activities were detected during the last iteration, the algorithm is terminated.

The iterative procedure provides a new way to eliminate the need to know in advance the true number of activities. In every step we estimate the number of positive and negative activities. This number can be used in the noise level estimator for improving the accuracy. If the initial conditions are set correctly, this procedure will stops. We provide here an intuitive explanation how this procedure works:

We set the initial condition to be $\tilde{N} = N$, that is - no activities. In the first iteration the noise estimate will have positive bias, which will decrease the probability of false alarm (compared to the planned one). The strong activities will be detected while the weak activities will remain undetected. In the next iteration the noise level estimate will be smaller than the noise estimate in the previous iteration. This decrease will lower the threshold which, in turn, causes more activities to be detected. This process will continue until it stops when no more activities will be detected. Does it happen? the answer is yes. Assume $(w.l.g.)$ that no activities exist. We start the process and if the probability of false alarm is very small compared to the number of measurements, than no more activities will be detected and the process will stop. If, however, the probability of false alarm is much greater than the number of measurements, few noise measurements will be detected as activities.

## 5. SIMULATIONS

To demonstrate the performance of the proposed algorithm, consider the following example: $N = 1024, s_1 = 10, s_2 = 5, s_3 = 1, s_i = 0, i > 3, \sigma^2 = 1$. The following table depicts the probability of detection as a function of the designed probability of false alarm for the three possible bins, $s_1, s_2, s_3$. We have simulated algorithms I and II and computed their probability of detection and false alarm We denote by $P_D(s_i)_{Alg\ j}$ the probability of detection of $s_i$ by the $j$-th algorithm. For simplicity, the threshold (i.e., $a$) has been set assuming the noise level is *known*.

In the first experiment we used the true number of sources for the noise level estimator in Algorithm I. The initial condition for Algorithm II was $N = \tilde{N}$ (no activities). The results are given in the next table.

| Designed $P_{fa}$ | $1e^{-4}$ | $5e^{-4}$ | $1e^{-3}$ | $5e^{-3}$ |
|---|---|---|---|---|
| Empirical $P_{fa}$ | $1.4e^{-4}$ | $6.4e^{-4}$ | $1.25e^{-4}$ | $5.63e^{-3}$ |
| $P_D(s_1)_{Alg1}$ | 1 | 1 | 1 | 1 |
| $P_D(s_2)_{Alg1}$ | 0.61 | 0.77 | 0.83 | 0.93 |
| $P_D(s_3)_{Alg1}$ | 0 | 0.01 | 0.02 | 0.07 |
| $P_D(s_1)_{Alg2}$ | 1 | 1 | 1 | 1 |
| $P_D(s_2)_{Alg2}$ | 0.59 | 0.76 | 0.82 | 0.93 |
| $P_D(s_3)_{Alg2}$ | 0 | 0.01 | 0.02 | 0.06 |

It shows that the performance of the two algorithms is similar, and - in particular - the empirical probability of false alarm is almost equal to the designed probability of false alarm.

However, when in Algorithm I a wrong number of activities is assumed, the situation is different. In the second experiment 10 activities were assumed when using the noise level estimator in the first algorithm.

| Designed $P_{fa}$ | $1e^{-4}$ | $5e-4$ | $1e^{-3}$ | $5e^{-3}$ |
|---|---|---|---|---|
| Empirical $P_{fa}$ | $2.7e^{-4}$ | $9.6e^{-4}$ | $1.73e^{-3}$ | $7.09e^{-3}$ |
| $P_D(s_1)_{Alg1}$ | 1 | 1 | 1 | 1 |
| $P_D(s_2)_{Alg1}$ | 0.66 | 0.80 | 0.85 | 0.94 |
| $P_D(s_3)_{Alg1}$ | 0.01 | 0.02 | 0.03 | 0.08 |

While the probability of detection is unchanged, the designed probability of false alarm differs considerably from the empirical probability of false alarm.

This experiment demonstrates the sensitivity of the performance of algorithm I to the prior knowledge on the number of activities. Algorithm II, however, is a CFAR detector even where the noise level and the number of activities are unknown.

## 6. DISCUSSION

The key point in our proposed algorithm is the improved noise level estimate, which can be achieved even if the number of activities is unknown. It is based on an iterative censoring based procedure. However, using censored sample to eliminate activities is not a new idea and it has been previously used for detecting known signal in unknown noise level environment [5].

In radar systems the problem of detecting a signal which is known up to a phase term in Gaussian noise of unknown level is an essential one. Based on the common implementation, this problem is usually modeled as follows: Let $x_1, \ldots, x_N$ be an *exponential* random variable, where the mean of $x_2, \ldots, x_N$ is equal to $\sigma^2$ and the mean of $x_1$ is equal to $\sigma^2 + s$. If $s = 0$ then no signal is present, if $s > 0$ the signal is present. Usually, $x_1$ is the received match filtered signal and $x_i$ $i \geq 2$ are usually samples of the match filtered noise. The optimal invariant detector for such a problem is given by:

$$T(x_1, \ldots, x_N) = \frac{x_1}{\frac{\sum_{i=2}^{N} x_i}{N-2}} \mathop{\gtrless}_{Noise}^{Signal+Noise} \delta$$

As pointed out in [9] (among many others), the performance of this detector decreases when the noise measurements, $x_2, \ldots, x_N$ contain interference. To avoid this difficulty it was suggested to censor few of the largest valued measurements and to estimate the noise level based on censored samples. Such a detector is usually refereed to as the OS-CFAR detector, which can be found in radar quite extensively [7].

The algorithm presented here suggest a new approach to deal with a much larger class of problems, where the noise level is *unknown*. Our approach can be regarded as an extension of the $OS-CFAR$ detector. The proposed algorithm is, to the best of our knowledge, the first CFAR detector for the case where both the noise level and the number of activities are unknown. As such, it is flexible in the sense that it provides means to control the performance of the algorithm (i.e., the probability of false alarm).

In [3] we suggest new estimation procedures for the normal model and other probabilistic models. We also give exact expressions for the probability of false alarm as a function of the threshold. The analysis is carried out for few important model.

## 7. REFERENCES


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## 8. APPENDIX

The Taylor series for $\frac{f(z_{r+1:N})}{F(z_{r+1:N})}$ and $\frac{f(z_{n-s:N})}{1-F(z_{N-s:N})}$ around the points $\xi_{r+1} = F^{-1}(p_{r+1})$ and $\xi_{N-s} = F^{-1}(p_{N-s})$ are given by the following equations:

$$\frac{f(z_{r+1:N})}{F(z_{r+1:N})} \approx \alpha - \beta z_{r+1:N} \quad (9)$$

$$\frac{f(z_{N-s:N})}{1-F(z_{N-s:N})} \approx \gamma + \delta z_{N-s:N} \quad (10)$$

where

$$\alpha = f(\xi_{r+1})\{1 + \xi_{r+1}^2 + \xi_{r+1} f(\xi_{r+1})/p_{r+1}\}/p_{r+1} \quad (11)$$

$$\beta = f(\xi_{r+1})\{f(\xi_{r+1}) + p_{r+1}\xi_{r+1}\}/p_{r+1}^2 \quad (12)$$

$$\gamma = f(\xi_{n-s})\{1 + \xi_{n-s}^2 - \xi_{n-s} f(\xi_{n-s})/q_{n-s}\}/q_{n-s} \quad (13)$$

$$\delta = f(\xi_{n-s})\{f(\xi_{n-s}) - q_{n-s}\xi_{n-s}\}/q_{n-s}^2 \quad (14)$$

The ratio $\frac{f'_X(x)}{f_X(x)}$ is equal to $-x$ when $X$ is the standard normal random variable. Submitting back these expressions (11,12,13,14) and the ratio into equation (5) results in the approximated equation given by:

$$\frac{\partial \ln L(\sigma)}{\partial \sigma} \approx -\frac{1}{\sigma}\left[A + r\alpha z_{r+1:N} - s\gamma z_{N-s:N} - \right.$$
$$\left. -r\beta z_{r+1:N}^2 - s\delta z_{N-s:N}^2 - \sum_{i=r+1}^{N-s} z_{:N} i^2 \right] = 0 \quad (15)$$

The solution of (15) is given by the following equation:

$$\hat{\sigma} = \frac{-D + \sqrt{D^2 + 4AE}}{2A} \quad (16)$$

where

$$D = r\alpha x_{r+1:N} - s\gamma x_{N-s:N} \quad (17)$$

$$E = \sum_{i=r+1}^{N-s} x_{i:N}^2 + r\beta x_{r+1:N}^2 + s\delta x_{N-s:N}^2 \quad (18)$$