# DESIGN OF LINEAR PHASE IIR FILTERS VIA WEIGHTED LEAST-SQUARES APPROXIMATION


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## ABSTRACT

A new method for designing IIR digital filters with linear phase in the passband is proposed. This method is based on frequency weighted least-square error optimization using the BFGS method [15]. The gradient of the cost function with respect to the design parameters, required for the implementation of the BFGS method, is derived. The proposed method is started by obtaining an initial IIR filter design using model reduction of a linear phase FIR filter. Based on this initial design the cost function is minimized using the BFGS method. An example shows that the proposed method leads to very good filter designs.


## 1. INTRODUCTION

The design of infinite impulse response (IIR) filters with linear phase in the passband has been considered by many researchers in the last two decades. The problem to be solved is the simultaneous approximation of both magnitude and phase characteristics. The conventional technique is to first design an IIR filter that meets the magnitude specifications and then to employ all-pass equalizers to linearize the phase response [1], [2]. Another technique is using linear programming to directly design IIR filters which meet both magnitude and phase characteristics simultaneously [3]. Other methods for designing linear phase IIR filters using model reduction have been proposed in the literature, see [4]-[9] and the references therein. In [10], Holford and Agathoklis applied frequency weighting model reduction based on the balanced realization and the impulse response gramian to design linear phase IIR filters from linear phase FIR filters. This technique is using input frequency weighting to improve the approximation in the transition region. For highly selective filters, it was shown by numerical examples that the method in [10] gives good compromise for the filter order, passband maximum ripple and stopband minimum attenuation. In [11], Brandenstein and Unbehauen presented an algorithm for the least-squares approximation of FIR filters by IIR filters. The examples showed that the approximation error of the method in [11] is as small as that of the IIR filters obtained with the balanced model reduction [8] and the impulse response gramian model reduction [7]. In [12], Lu, Pei and Tseng proposed a weighted least-squares algorithm for the design of IIR filters including linear phase IIR filters. This algorithm is directly designing IIR filters rather than approximating FIR filters, and is based on an iterative scheme. In [13], Li, Xie, Yan and Soh employed orthogonal projection to improve the design of IIR filters.

In this paper, a new method for designing approximately linear phase IIR filters is presented. An initial IIR filter design is obtained from a linear phase FIR filter satisfying the design specifications. In most cases, frequency specifications are violated first in the transition band during model reduction [10]. To improve the approximation and obtain a low order IIR filter which still satisfies the design specifications, a frequency weighted error function is optimized. This error function between the original specifications and the frequency response of the IIR design is minimized using the Broyden-Flether-Goldfarb-Shanno (BFGS) method [15], [16], which is the 'best' quasi-Newton optimization method. It is more efficient than the steepest descent method, the conjugate gradient methods and DFP method. This BFGS method requires the computation of the gradient of the error function which can be derived based on the solution of two Lyapunov equations. In the next section the problem is formulated and then the design algorithm is presented in section 3. An example in section 4 illustrates the method developed and shows that it leads to very good designs.

## 2. PROBLEM FORMULATION

Consider a stable, reachable and observable, linear time-invariant, $m$th-order digital system described by the following state-space model

$$x(i+1) = Ax(i) + bu(i) \tag{1a}$$
$$y(i) = cx(i) + du(i) \tag{1b}$$

with $A \in R^{m\times m}$, $b \in R^{m\times 1}$, $c \in R^{1\times m}$ and $d \in R$. The system transfer function can be expressed in terms of the constant matrices as

$$H(z) = c(zI - A)^{-1}b + d \tag{2}$$

The weighted least-squares approximation of $H(z)$ is to find, for a given $n < m$, a reduced order system with the transfer function

$$H_r(z) = c_r(zI - A_r)^{-1}b_r + d \tag{3}$$

where $A_r \in R^{n\times n}$, $b_r \in R^{n\times 1}$ and $c_r \in R^{1\times n}$, which minimizes the following cost function

$$J_{ew} = \|(H_r(z) - H(z))H_w(z)\|_2 \tag{4a}$$
$$= \left(\frac{1}{2\pi j}\oint_{|z|=1} |(H_r(z) - H(z))H_w(z)|^2 \frac{dz}{z}\right)^{\frac{1}{2}} \tag{4b}$$

where $H_w(z)$ is a given weighting function described by

$$H_w(z) = c_w(zI - A_w)^{-1}b_w + d_w \tag{5}$$

with $A_w \in R^{k\times k}$, $b_w \in R^{k\times 1}$, $c_w \in R^{1\times k}$ and $d_w \in R$.

Let $H_{ew}(z)$ be the weighted error function given by $H_{ew}(z) = (H_r(z) - H(z))H_w(z)$, then $H_{ew}(z)$ can be expressed by

$$H_{ew}(z) = C_{ew}(zI - A_{ew})^{-1}B_{ew} \tag{6}$$

with

$$A_{ew} = \begin{bmatrix} A_r & 0 & b_r c_w \\ 0 & A & bc_w \\ 0 & 0 & A_w \end{bmatrix}, \quad B_{ew} = \begin{bmatrix} b_r d_w \\ bd_w \\ b_w \end{bmatrix} \tag{7a}$$
$$C_{ew} = \begin{bmatrix} c_r & -c & 0 \end{bmatrix}. \tag{7b}$$

Based on (4), (6) and (7), it is not difficult to show that $J_{ew}$ can be expressed by

$$J_{ew} = \left(C_{ew}K_{ew}C_{ew}^T\right)^{\frac{1}{2}} \tag{8a}$$
$$= \left(B_{ew}^T W_{ew} B_{ew}\right)^{\frac{1}{2}} \tag{8b}$$

where $K_{ew}$ and $W_{ew}$ are the solutions to the following two Lyapunov equations

$$K_{ew} - A_{ew}K_{ew}A_{ew}^T = B_{ew}B_{ew}^T \tag{9}$$
$$W_{ew} - A_{ew}^T W_{ew} A_{ew} = C_{ew}^T C_{ew}. \tag{10}$$

The problem considered here is minimizing $J_{ew}$ with respect of the coefficients of the reduced order model $A_r$, $b_r$ and $c_r$. This will be carried out usig the BFGS method[15].

## 3. WEIGHTED LEAST-SQUARES APPROXIMATION

The BFGS method [15], [16] requires the computation of the gradient of the cost functional $J_{ew}$ for the implementation. This is presented first, and then the linear phase IIR filter design algorithm is presented.

### 3.1. Gradient of the Cost Function

**Theorem 1:** The gradient of the cost function $J_{ew}$ with respect to $A_r$, $b_r$ and $c_r$ is given by

$$\frac{\partial J_{ew}}{\partial A_r} = 2\begin{bmatrix} I_n & 0 \end{bmatrix} W_{ew}A_{ew}K_{ew}\begin{bmatrix} I_n & 0 \end{bmatrix}^T \tag{11}$$
$$\frac{\partial J_{ew}}{\partial b_r} = 2\begin{bmatrix} I_n & 0 \end{bmatrix} W_{ew}A_{ew}K_{ew}\begin{bmatrix} 0 & c_w \end{bmatrix}^T + 2\begin{bmatrix} I_n & 0 \end{bmatrix} W_{ew}B_{ew}d_w \tag{12}$$
$$\frac{\partial J_{ew}}{\partial c_r} = 2C_{ew}K_{ew}\begin{bmatrix} I_n & 0 \end{bmatrix}^T \tag{13}$$

*Proof:* This theorem can be proven based on the matrix differential rules presented in [14]. Details are omitted here for brevity.

The gradient of $J_{ew}$ can thus be computed using the solutions to the two Lyapunov equations (9) and (10) and the above equations.

### 3.2. Linear Phase IIR filter design algorithm

The IIR design algorithm can now be presented. The BFGS will be used to minimize $J_{ew}$ and the gradient of $J_{ew}$ will be computed using Theorem 1.

**Algorithm**: Linear phase IIR filter design method.

1. Design a linear phase FIR filter $H(z)$ satisfying the given specifications with

$$H(z) = \sum_{i=0}^{m} h_i z^{-1} \tag{14}$$

2. Realize the FIR filter $H(z)$ using a state-space model as in (1) where

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{15a}$$

$$c = \begin{bmatrix} h_1 & h_2 & \cdots & h_m \end{bmatrix}, d = h_0. \tag{15b}$$

3. Conduct balanced model reduction [17]-[19] on $H(z)$ (2) with (15) to get a good initial solution $(A_r, b_r, c_r, d)$, then transform it to controllable canonical form with $b_r = [10 \cdots 0]^T$.

4. Select a suitable frequency-weighted function $H_w(z)$ and express it as (5) to get $(A_w, b_w, c_w, d_w)$.

5. Formulate the weighted error function $H_{ew}(z)$ in the form of equations (6) and (7).

6. Calculate the gradients of the cost function according to (11)–(13).

7. Based on BFGS algorithm [15], [16] minimize the cost function $J_{ew}$ to obtain an optimal solution $(A_r, b_r, c_r, d)$.

The obtained $(A_r, b_r, c_r, d)$ at step 7 is then an IIR filter with linear phase in the passband. In the appendix, the BFGS algorithm is given in detail. The choice of the frequency weighting function is similar to the method given in [20].

## 4. EXAMPLE

In this section, an example is presented to show that the proposed method provides very good design results.

A low-pass FIR filter of order 60, whose amplitude response and phase response are shown in Figure 1, has been reduced to 12th order IIR filter whose amplitude response and phase response are shown in Figure 2.

Clearly the amplitude response of the IIR filter is very close to the one of the FIR filter, and the phase response of the IIR filter is linear in the passband. It should be pointed out that the order of the resulting IIR filter is only one fifth of the order of the original FIR filter. This is lower than many other techniques for linear phase IIR filter design based on model reduction.

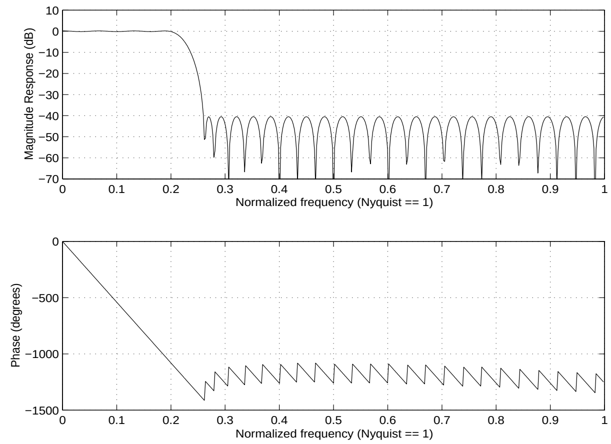


Fig. 1, the original 60th-order FIR filter

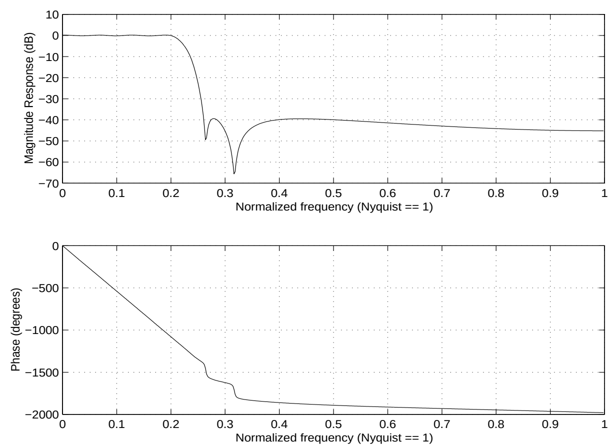


Fig. 2, the order-reduced 12th-order IIR filter

## 5. CONCLUSION

In this paper, a new method for designing linear phase IIR filters is proposed. This method is based on frequency weighted least-square error optimization using the BFGS method. The gradient of the cost function with respect to the design parameters, required for the implementation of the BFGS method, is derived. The proposed method begins with obtaining an initial IIR filter design using model reduction of a linear phase FIR filter. Based on this initial design the weighted least square cost function is minimized using the BFGS method. An example shows that the proposed method leads to very good filter design results.

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## 7. APPENDIX: BFGS ALGORITHM

Initialization

$$w^{(0)} = [A_r(1{,}1{:}\mathrm{n})]^T, \quad g^{(0)} = \left[\frac{\partial J_{ew}}{\partial A_r}(1{,}1{:}\mathrm{n})\right]^T$$

$$P^{(0)} = I_{n\times n}, \qquad \epsilon = 10^{-5},\ k=0,\ L=10$$

Iteration

**FOR** k=0:L

If $\|\frac{\partial J_{ew}}{\partial A_r}\|_2 < \epsilon$, then **Stop**

$s^{(k)} = -P^{(k)} g^{(k)}$

line search along $s^{(k)}$ giving $w^{(k+1)} = w^{(k)} + \alpha^{(k)} s^{(k)}$

update $A_r$ by $A_r(1{,}1{:}\mathrm{n}) = [w^{(k+1)}]^T$

recalculate $K_{ew}$ with updated $A_r$ in $A_{ew}$

update $c_r = c * K_{ew}(\mathrm{n+1{:}n+m,1{:}n}) * inv[K_{ew}(1{:}\mathrm{n},1{:}\mathrm{n})]$

recalculate $W_{ew}$ with updated $c_r$ in $C_{ew}$, update $J_{ew}$

$$g^{(k+1)} = \left[\frac{\partial J_{ew}}{\partial A_r}(1{,}1{:}\mathrm{n})\right]^T$$

$$\delta^{(k)} = w^{(k+1)} - w^{(k)}, \qquad \gamma^{(k)} = g^{(k+1)} - g^{(k)}$$

$$P^{(k+1)} = P^{(k)} + \left[1 + \frac{\gamma^{(k)T} P^{(k)} \gamma^{(k)}}{\delta^{(k)T}\gamma^{(k)}}\right] \frac{\delta^{(k)}\delta^{(k)T}}{\delta^{(k)T}\gamma^{(k)}} - \left[\frac{\delta^{(k)}\gamma^{(k)T} P^{(k)} + P^{(k)}\gamma^{(k)}\delta^{(k)T}}{\delta^{(k)T}\gamma^{(k)}}\right]$$

**END**