Optimal Array Pattern Synthesis Using Semidefinite Programming

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Abstract

In this paper, we present a new technique to solve array pattern synthesis problems by using semidefinite programming. We first formulate (or reformulate) the array design problems into semidefinite programming problems, and then use the recently developed efficient numerical algorithms and software to compute the numerical solution of antenna array weights. Using this approach, we can directly solve not only the standard synthesis problems for nonuniform arrays, but also the synthesis problems for arrays having power restrictions and uncertainties. Numerical examples are presented to illustrate our approach.

I. Introduction

Antenna arrays have many applications in signal processing and communications. A basic problem of antenna arrays is to arrange array weights to obtain higher directive gain, or spatially filtering signals that do not come from the desired direction.

For uniform linear array, a celebrated solution was given by Dolph [1] using the Chebyshev polynomial. The designed pattern, which is known as the Dolph-Chebyshev pattern, has the property of the minimum sidelobe level with a given mainlobe width.

However for arbitrary arrays, the Dolph-Chebyshev method can not be applied. To optimally design weights for arrays containing nonuniformly spaced or nonequal elements, or for arrays with nonlinear shape, several algorithms were developed by formulating synthesis problems as quadratic programming problems (see for example [2, 3, 4]), where the sum of the squared synthesis error between the synthesized pattern and the desired pattern is minimized.

Another approach for array pattern synthesis is based on the concept of artificial interference. Applying the adaptive array theory, [5, 6] design antenna arrays by maximizing the signal-interference-noise-ratio (SINR).

Since in many cases we are indeed interested in minimizing the peaks of the synthesis error, one step of optimization may not give a pattern of desired quality. In [2, 7], array synthesis algorithms to minimize a quadratic objective function with recursively updated linear constraints were proposed. Another way to minimize the peaks of the synthesis error is to add a weighting function in the objective function or recursively adjusting the interference-noise-ratio (INR) [6]. In [8], a numerically more efficient algorithm by using recursive least square method was proposed.

By observing that the array pattern synthesis problem is indeed a convex optimization problem, [9] proposed to solve the single look direction array pattern synthesis problem using the interior point method.

In this paper, we present a new approach to *directly* solve the array pattern synthesis problems. We first formulate (or reformulate) the array pattern synthesis problems as a special class of convex optimization problems — the semidefinite programming (SDP) problems or linear objective minimization problems with linear matrix inequality (LMI) constraints. In general, these LMI problems can not be solved analytically. However, they can be numerically solved very efficiently by using the recently developed numerical algorithms and software. Therefore, the "solution" of the array synthesis problems presented in this paper will be linear objective optimization problems with linear matrix inequalities constraints. Here the "solution" is in the sense that they can be directly solved using the optimization software, for example the Matlab LMI Toolbox [10].

In comparison to the conventional methods, the SDP approach leads to a *direct optimal* solution to the array pattern synthesis problem. In addition, this approach can also solve the synthesis problems when the arrays have additional constraints, such as power restrictions or uncertainties. Here we note that the semidefinite programming techniques are applied for statistically optimal beam forming problems in [11].

Our notations are standard. $P > 0$ ($P \geq 0$) means that P is a real, symmetric, positive-definite (semi-positive-definite) matrix. For brevity, we omit all proofs and only show one numerical example; a more complete version of the paper can be obtained from the authors via email `fanw@cig.mot.com`.

II. Optimal Array Pattern Synthesis

A. Problem setup

Consider a linear array with N antennas. Assuming that the arriving signal is a narrow band signal, we denote the

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steering vector as

$$V(\theta) = \left[\begin{array}{ccc} f_1(\theta)e^{j\phi_1(\theta)} & \cdots & f_N(\theta)e^{j\phi_N(\theta)} \end{array} \right]^T,$$

where $f_i(\theta)$ is the element pattern of the i-th antenna, θ is the angle of the arrival signal. $\phi_i(\theta)$ is the phase delay due to propagation, and for a linear array can be represented as

$$\phi_i(\theta) = \frac{2\pi d_k \sin\theta}{\lambda}, \qquad (1)$$

where λ is the wave length of the transmitted signal, d_k is the position of the k-th element. Then the output of the antenna array is represented as

$$P(\theta) = \sum_{i=1}^{N} w_i f_i(\theta) e^{j\phi_i(\theta)} = W^T V(\theta), \qquad (2)$$

where $W = [w_1 \ \cdots \ w_N]^T \in \mathbf{C}^N$ is the complex weight vector of the array.

For the antenna array described above, an array pattern synthesis problem can be stated as

$$\begin{array}{l} \textit{For given } \epsilon(\theta) > 0, \textit{ find if it exists,} \\ \textit{a complex weight vector } W, \textit{ such that} \\ |P(\theta) - P_{\rm d}(\theta)| \le \epsilon(\theta), \forall\theta \in \mathbf{\Theta}, \end{array} \qquad (3)$$

where $P_{\rm d}(\theta)$ is the desired pattern, and $\mathbf{\Theta}$ is the set of arrival angles that we are of interest. Normally, $\mathbf{\Theta}$ is a "dense set" of the interval $[-90, 90]$. Thus we require the synthesized pattern approach to the desired pattern in the whole range of arrival angles from $-90°$ to $90°$.

$\epsilon(\theta)$ is the allowed synthesis error at the arrival angle θ. Let $\epsilon(\theta) = r(\theta)\epsilon$, where $r(\theta)$ is a predefined ratio of angular response of antenna array at different arrival angles. For example, $r(\theta)$ can be the ratio of the magnitude of the ripples in the main beam and the magnitude of the ripples in the side lobe [8]. By minimizing the level of the synthesis error ϵ in (3), we can achieve the minimax pattern.

Since we do not have any restriction on the steering vector $V(\theta)$, the problem we consider is a general array pattern synthesis problem, where different array elements can have different element patterns, and the array can have nonuniform spacing between elements.

By observing the convexity of the array pattern synthesis problems, Lebret and Boyd proposed to solve the single look direction array pattern synthesis problems using the interior point method [9]. With the same observation, we formulate the array pattern synthesis problems into linear matrix inequality (LMI) problems, and solve them using the recently developed software and algorithms. In comparing with the conventional recursive methods, our method has the following advantages:

1. Our method is based on semidefinite programming. Therefore we can take advantage of the recently developed algorithms and software and the problem can be numerically solved efficiently;
2. Since most LMI algorithms (software) solve the original optimization problem and its dual problem simultaneously, our method can give a determined answer for the "infeasibility" of the problem (3), i.e., for the given $\epsilon(\theta)$, problem (3) is infeasible for any $W \in \mathbf{C}^N$;
3. By using the semidefinite programming approach, we can have a "guaranteed accuracy[1]" of the optimal solution of the array weights. The optimality and accuracy of the solutions by using conventional recursive methods normally can not be guaranteed.
4. Our method can not only solve the basic array pattern synthesis problem (3), but also design optimal array patterns when the array is power limited or has uncertainties.

B. *Array pattern synthesis using LMI approach*

Theorem 2.1 *For a given weight vector $W \in \mathbf{C}^N$, the condition*

$$|P(\theta) - P_{\rm d}(\theta)|^2 \le \epsilon(\theta) \qquad (4)$$

holds if and only if the following linear matrix inequality holds

$$\left[\begin{array}{cc} \epsilon(\theta) + 2\hat{W}^T U(\theta) R(\theta) - R(\theta)^T R(\theta) & \hat{W}^T U(\theta) \\ U(\theta)^T \hat{W} & I \end{array} \right] \ge 0 \qquad (5)$$

where $\hat{W}$ is the optimization variable and

$$\hat{W} = \left[\begin{array}{c} W_R \\ W_I \end{array} \right], \ W_R = \mathrm{Re}\{W\}, \ W_I = \mathrm{Im}\{W\}, \qquad (6a)$$

$$U(\theta) = \left[\begin{array}{cc} V_R(\theta) & V_I(\theta), \\ -V_I(\theta) & V_R(\theta) \end{array} \right], R(\theta) = \left[\begin{array}{c} P_{{\rm d},R}(\theta) \\ P_{{\rm d},I}(\theta) \end{array} \right], \qquad (6b)$$

$$V_R(\theta) = \mathrm{Re}\{V(\theta)\}, \ V_I(\theta) = \mathrm{Im}\{V(\theta)\}, \qquad (6c)$$

$$P_{{\rm d},R}(\theta) = \mathrm{Re}\{P_{\rm d}(\theta)\}, \ P_{{\rm d},I}(\theta) = \mathrm{Im}\{P_{\rm d}(\theta)\}. \qquad (6d)$$

Theorem 2.1 is a main result of this paper. With Theorem 2.1, we can formulate the array pattern synthesis problem (3) as an LMI feasibility problem:

$$\begin{array}{l} \textit{For given } \epsilon(\theta), \textit{ find a weight factor } W, \\ \text{such that LMI (5) holds for all } \theta \in \mathbf{\Theta}. \end{array} \qquad (7)$$

This problem can be solved efficiently by using the standard numerical algorithms, for example the Matlab LMI Toolbox [10]. Since the solver of LMI is based on the primal-dual interior point method, it can give not only a feasible solution when (7) is feasible for the given $\epsilon(\theta)$, but also a determined answer for the infeasibility when (7) is not feasible for any $W \in \mathbf{C}^N$ with the given $\epsilon(\theta)$.

III. Specific problems

In Section II, we have derived the basic LMI synthesis conditions for arrays whose elements are nonuniformly spaced, or the element patterns are nonequal. In this section, we consider several specific design problems for such arrays:

1. Minimize the error between the synthesized pattern and the desired pattern, or minimax array pattern synthesis problem;

[1] Since the solver of LMI problems are based on the primal-dual method, the relative accuracy required on the optimal value of the objective can be guaranteed.

2. Design optimal arrays with power restriction;
3. Design worst case performance guaranteed patterns for arrays having uncertainties, or robust array pattern synthesis problem.

We will show that these problems can also be formulated as semidefinite programming problems and solved using the standard algorithms.

A. Minimax array pattern synthesis

In some cases, we wish to minimize the sidelobe level. More generally, we may wish to minimize the maximum synthesis error between the synthesized pattern and the desired pattern, so that the synthesized pattern is as close to the desired pattern as possible. This problem can be easily formulated as a convex optimization problem with LMI constraints:

$$\texttt{minimize}\ \epsilon, \quad \texttt{subject to}\ \text{LMI (5) for all}\ \theta \in \mathbf{\Theta}. \tag{8}$$

Here we recall that $\epsilon(\theta) = r(\theta)\epsilon$. In the following, we assume $r(\theta) = 1$ for the simplicity of the description. The results established can be easily extended to general $r(\theta)$.

B. Array pattern synthesis with power restriction on weight vector

Because of the power restriction of the array network [12], we may have an addition condition on the weight vector that

$$\sum_{i=1}^{N} |w_i|^2 \leq \gamma, \tag{9}$$

where $\gamma > 0$ is the restriction of the power. Note that $\|W\|^2 = \sum_{i=1}^{N} |w_i|^2$. This power restriction (9) can be represented as the following equivalent LMI condition

$$\begin{bmatrix} \gamma & \hat{W}^T \\ \hat{W} & I \end{bmatrix} \geq 0, \tag{10}$$

where $\hat{W}$ is defined in (6).

To synthesize a array pattern satisfying the power restriction (10), we only need to solve the following LMI problem:

For a given ϵ and the power restriction γ, find W, such that LMIs (5) and (10) hold for all $\theta \in \mathbf{\Theta}$. (11)

C. Array pattern synthesis with output power restriction

In the array pattern synthesis problem (3), the performance requirement is on each arrival angle in the set $\mathbf{\Theta}$. In some cases, we may have a requirement on the array output power within a range of arrival angles, i.e.,

$$\sum_{\theta \in \mathbf{\Phi}} |P(\theta)|^2 \leq \xi, \tag{12}$$

where $\mathbf{\Phi} = \{\theta_1, \cdots, \theta_m\}$.

Define

$$\mathcal{V} = [\ V(\theta_1) \quad \cdots \quad V(\theta_m)\ ], \mathcal{U} = \begin{bmatrix} \text{Re}\{\mathcal{V}\} & \text{Im}\{\mathcal{V}\} \\ -\text{Im}\{\mathcal{V}\} & \text{Re}\{\mathcal{V}\} \end{bmatrix}.$$

Then (12) is equivalent to $W^T \mathcal{V}\mathcal{V}^T W \leq \xi$. It is further equivalent to an LMI condition

$$\begin{bmatrix} \xi I & \hat{W}^T \mathcal{U} \\ \mathcal{U}^T \hat{W} & I \end{bmatrix} \geq 0. \tag{13}$$

We note that condition (13) can be combined with the results in Section III-A and Section III-B if we have both the requirement (4) on the arrival angles in $\mathbf{\Theta}$ and the requirement (12) on the arrival angles in $\mathbf{\Phi}$.

D. Robust array pattern synthesis

D.1 Element gain uncertainty

Due to the nonlinearity and uncertainties of the amplifiers in the array, the array element may suffer from uncertainties in amplification gain. We consider the following array with gain uncertainties:

$$V_\delta(\theta) = \left[\ (1+\delta) f_1(\theta) e^{j\phi_1(\theta)} \quad \ldots \quad f_N(\theta) e^{j\phi_N(\theta)}\ \right]^T, \tag{14}$$

where δ is a uncertainty and satisfies $|\delta| \leq \rho$. ρ is a known bound.

The robust array pattern requires

$$\max_{|\delta| \leq \rho} |P_\delta(\theta) - P_{\text{d}}(\theta)|^2 \leq \epsilon, \tag{15}$$

for all θ in $\mathbf{\Theta}$, where $P_\delta(\theta) = W^T V_\delta(\theta)$.

For an array with uncertain element described in (14), we have the following result on the robust pattern synthesis.

Theorem 3.1 *For a given weight vector $W \in \mathbf{C}^N$, the condition (15) holds if and only if the following linear matrix inequalities hold*

$$\begin{bmatrix} \epsilon + 2\hat{W}^T \overline{U}(\theta) R(\theta) - R(\theta)^T R(\theta) & \hat{W}^T \overline{U}(\theta) \\ \overline{U}(\theta)^T \hat{W} & I \end{bmatrix} \geq 0, \tag{16a}$$

$$\begin{bmatrix} \epsilon + 2\hat{W}^T \underline{U}(\theta) R(\theta) - R(\theta)^T R(\theta) & \hat{W}^T \underline{U}(\theta) \\ \underline{U}(\theta)^T \hat{W} & I \end{bmatrix} \geq 0, \tag{16b}$$

where $\hat{W}$ is the optimization variable and

$$\overline{U}(\theta) = \begin{bmatrix} \overline{V}_R(\theta) & \overline{V}_I(\theta) \\ -\overline{V}_I(\theta) & \overline{V}_R(\theta) \end{bmatrix}, \underline{U}(\theta) = \begin{bmatrix} \underline{V}_R(\theta) & \underline{V}_I(\theta) \\ -\underline{V}_I(\theta) & \underline{V}_R(\theta) \end{bmatrix},$$

and

$$\overline{V}_R(\theta) = \text{Re}\{V_\rho(\theta)\}, \quad \overline{V}_I(\theta) = \text{Im}\{V_\rho(\theta)\},$$
$$\underline{V}_R(\theta) = \text{Re}\{V_{-\rho}(\theta)\}, \quad \underline{V}_I(\theta) = \text{Im}\{V_{-\rho}(\theta)\},$$

$\hat{W}$ and $R(\theta)$ are defined in (6).

D.2 Element phase uncertainties

We now consider phase uncertainties in the steering vector, which may due to phase uncertainties in the propagation channel or uncertain positions of array elements.

Again, we assume that the uncertainties lie in the first array element, and the steering vector of the array can be represented as

$$V_\psi(\theta) = \left[\begin{array}{ccc} f_1(\theta)e^{j\phi_1(\theta)+j\psi} & \dots & f_N(\theta)e^{j\phi_N(\theta)} \end{array} \right]^T, \tag{17}$$

where ψ is a phase uncertainty and satisfies $|\psi| \leq \eta$. η is a known bound.

The robust array requires

$$\max_{|\psi|\leq\eta} |P_\psi(\theta) - P_{\rm d}(\theta)|^2 \leq \epsilon, \tag{18}$$

for all θ in $\mathbf{\Theta}$, where $P_\psi(\theta) = W^T V_\psi(\theta)$.

Following the same line as Theorem 3.1, we have the following result.

Theorem 3.2 *For a given weight vector $W \in \mathbf{C}^N$, the condition (18) holds if the following linear matrix inequalities hold for $i = 1, 2, 3, 4$*

$$\left[\begin{array}{cc} \epsilon + 2\hat{W}^T U_i(\theta)R(\theta) - R(\theta)^* R(\theta) & \hat{W}^T U_i(\theta) \\ U_i(\theta)^T \hat{W} & I \end{array} \right] \geq 0, \tag{19}$$

where $\hat{W}$ is the optimization variable and

$$U_i(\theta) = \left[\begin{array}{cc} V_{R,i}(\theta) & V_{I,i}(\theta) \\ -V_{I,i}(\theta) & V_{R,i}(\theta) \end{array} \right],$$

and

$$V_{R,i}(\theta) = {\rm Re}\{V_i(\theta)\}, \quad V_{I,i}(\theta) = {\rm Im}\{V_i(\theta)\},$$

$$V_i(\theta) = \left[\begin{array}{ccc} f_1(\theta)e^{j\phi_1(\theta)}K_i & \cdots & f_N(\theta)e^{j\phi_N(\theta)} \end{array} \right]^T,$$

$$K_1 = e^{j\eta}, K_2 = e^{-j\eta}, K_3 = 1 + j\sin\eta, K_4 = 1 - j\sin\eta, \tag{20}$$

$\hat{W}$ and $R(\theta)$ are defined in (6).

IV. Numerical Examples

In this section, we present a numerical examples to illustrate the algorithms proposed in this paper.

Consider a non uniformly spaced power restricted 41-element array from [8]. Because of the power limitation, we have a restriction on the weight vector

$$\sum_{i=1}^{41} |w_i|^2 \leq C. \tag{21}$$

The desired array pattern has a flap-top main beam. The region $|\theta| \leq 20°$ corresponds to the main beam, or pass band; the region $|\theta| \geq 25°$ corresponds to the sidelobe, or stopping band.

We first relax the power restriction (21). The optimized array pattern has a side lobe of about -30dB, see Figure 1. However, to achieve such a array pattern, the power of the weight vector is as high as 9.9005×10^8. Even though this design has the lowest sidelobe, it is not realizable due to its low radiation efficiency.

To design an array with high radiation efficiency, we include the power restriction condition (21). We observed

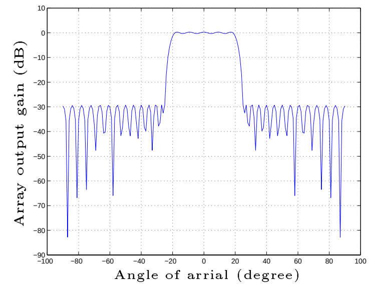

Fig. 1. Beam pattern with no power restriction

from Figure 2 that as a trade of tighter and tighter power restriction, the level of the error between the synthesized pattern and the desired pattern becomes larger and larger. With a power limitation of $\sum_{i=1}^{41} |w_i|^2 \leq 10$, the best array pattern has a sidelobe of -24.7 dB.

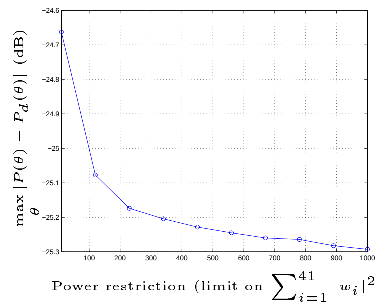

Fig. 2. Relation between the power restriction and the synthesis error

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