IF ESTIMATION OF HIGHER-ORDER POLYNOMIAL FM SIGNALS CORRUPTED BY MULTIPLICATIVE AND ADDITIVE NOISE

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ABSTRACT

In this paper, we propose the peak of the polynomial Wigner-Ville distribution as an instantaneous frequency estimator for polynomial frequency modulated signals in the presence of multiplicative and additive complex Gaussian processes. We show that this estimator is unbiased and we derive an analytic expression of its asymptotic variance. Simulation results, based on Monte-Carlo realisations, are presented in order to show the validity of the theoretical derivations.

1. INTRODUCTION

In many engineering applications such as radar, sonar, telecommunications and engine diagnosis, the signals under consideration are known to be non-stationary, i.e., their spectral contents vary with time. Time-frequency analysis, among other methods, was proposed to deal with such signals [1]. Time-frequency distributions (TFDs) are natural extensions of the Fourier transform. They map a one dimensional signal, function of time only, to a two dimensional quantity, function of time and frequency.

A concept intimately related to time-frequency distributions is that of instantaneous frequency (IF) [2]. In many situations, the IF characterises important physical parameters of the signal. For this, it is of great importance to have effective techniques to estimate the instantaneous frequency of a given signal.

Diverse IF estimation methods have been developed for *constant*, or slowly varying, amplitudes frequency modulated (FM) signals embedded in additive noise. Some of these methods are parametric and some are non-parametric. In general, parametric methods use a signal model and the goal, in this case, is to estimate some parameters in order to obtain the IF [3, 4, 5]. Non-parametric methods, on the other hand, use methods that do not require full knowledge of the signal. A well-known class of non-parametric methods, for IF estimation, is based on time-frequency distributions of the signal [6, 7].

In practice, in addition to the additive noise, the signal under consideration may be subjected to a *random* amplitude modulation which behaves as multiplicative noise [8]. Thus, it is important to have suitable IF estimation methods for such signals. In [9], we proposed the peak of the Wigner-Ville as an IF estimator for linear FM signals corrupted by multiplicative and additive noise. In this paper, we extend this result to the case of higher-order polynomial FM signals. The estimator considered here is based on the peak of the polynomial Wigner-Ville distributions (PWVD) [10]. The use of this particular class of time-frequency distributions stems from the fact that the PWVD yields maximum energy concentration, about the signal IF, for polynomial FM signals.

In particular, we derive analytical expressions for the bias and asymptotic variance of the IF estimator in the case of polynomial FM signals corrupted by multiplicative and additive complex circular Gaussian noise. We present some examples, using Monte-Carlo simulations, to show the validity of the derived theoretical results.

The paper is organised as follows. In Sections 2 and 3, we define the problem mathematically and give a brief review of the PWVD. In Section 4, we derive the analytical expressions of the bias and variance for the estimator. Some examples are presented in Section 5; whereas, Section 6 concludes the paper.

2. PROBLEM STATEMENT

Let the observed signal, $y(t)$, be given by

$$y(t) = m(t) \cdot z(t) + w(t) \tag{1}$$

where the stationary processes $m(t)$ and $w(t)$ are assumed circular complex Gaussian and independent with means and variances given by (μ_m, σ_m^2) and $(0, \sigma_w^2)$ respectively. The noiseless polynomial FM signal $z(t)$ is given by $z(t) = e^{j\phi(t)} = \exp\left\{j \sum_{i=0}^{P} a_i t^i\right\}$ where a_i are real coefficients and P is the order of the polynomial phase. Note that the derivation below does not require the knowledge of the coefficients a_i; it only assumes the signal to be a polynomial FM one.

Our primary interest is in estimating the instantaneous frequency of the signal $z(t)$ defined as

$$f_i(t) = \frac{1}{2\pi}\,\frac{d\phi(t)}{dt} = \exp\left\{ j \sum_{i=1}^{P} i a_i t^{i-1} \right\}$$

By writing the non-zero mean multiplicative noise $m(t)$ as $m(t) = \mu_m + m_o(t)$, with $m_o(t)$ being a zero-mean complex circular Gaussian noise process with variance σ_m^2, we can re-write the expression in Equation (1) as

$$\begin{aligned} y(t) &= \mu_m \cdot z(t) + m_o(t) \cdot z(t) + w(t) \\ &= |\mu_m| \cdot e^{j(\Theta_{\mu_m} + \phi(t))} + w_1(t) \end{aligned} \tag{2}$$

In Equation (2), $\Theta_{\mu_m} = \tan^{-1}\left[\frac{\text{Imag}(\mu_m)}{\text{Real}(\mu_m)}\right]$ and $w_1(t) = m_o(t) \cdot z(t) + w(t)$ is a zero-mean circular complex Gaussian noise with variance equal to $\sigma_{w_1}^2 = (\sigma_m^2 + \sigma_w^2)$. Note that the instantaneous frequencies of the signals $e^{j(\Theta_{\mu_m}+\phi(t))}$ and $e^{j\phi(t)}$ are exactly the same, i.e.,

$$f_i(t) = \frac{1}{2\pi} \cdot \frac{d(\Theta_{\mu_m} + \phi(t))}{dt} = \frac{1}{2\pi} \cdot \frac{d\phi(t)}{dt}.$$

We observe that, in this case, the problem of estimating the IF of the polynomial FM signal $z(t)$ affected by multiplicative and additive noise is equivalent to that of estimating the IF of a constant amplitude polynomial FM signal, $z_1(t) = |\mu_m| \cdot e^{j(\Theta_{\mu_m}+\phi(t))}$ (having the same IF law as $z(t)$), but affected by additive noise only.

3. PROPOSED ESTIMATOR

The polynomial Wigner-Ville distribution was designed to represent, in the time-frequency plane, polynomial FM signals as a row of delta functions about the signal IF. This class of time-frequency distributions is defined as [10]

$$W_z^{(q)}(t,f) = \int_{-\infty}^{\infty} \prod_{i=1}^{q/2} z(t + c_i\tau) z^*(t + c_{-i}\tau) e^{-j2\pi f\tau} d\tau \quad (3)$$

$$= \int_{-\infty}^{\infty} K_z^{(q)}(t,\tau) \cdot e^{-j2\pi f\tau} d\tau \quad (4)$$

where q is an even integer which indicates the order of non-linearity of the PWVD. The coefficients c_i and c_{-i} $(i = 1, 2, \ldots, q/2)$ are calculated so that the PWVD is real and equal to

$$W_z^{(q)}(n,f) = A^q \delta(f - f_i(t)),$$

for signals given by $z(t) = \exp\left\{j \sum_{i=0}^{P} a_i t^i\right\}$ (i.e., polynomial FM signals). The realness of the PWVD implies that $c_i = -c_{-i}$. Note that the Wigner-Ville distribution (WVD) is a member of the PWVDs class with parameters $q = 2$ and $c_1 = -c_{-1} = 0.5$. Full details of the design procedure may be found in [10].

The kernel of a particular PWVD, referred to as the sixth order PWVD, is given by [10]

$$\begin{aligned} K_z^{(6)}(t,f) &= [z(t + 0.62\tau)\, z^*(t - 0.62\tau)] \\ &\times [z(t + 0.75\tau)\, z^*(t - 0.75\tau)] \\ &\times [z(t - 0.87\tau)\, z^*(t + 0.87\tau)]. \end{aligned} \quad (5)$$

This distribution is optimal, in the sense of maximum signal energy concentration around the signal IF, for linear, quadratic and cubic FM signals $z(t) = \exp\left\{j \sum_{i=0}^{P} a_i t^i\right\}$ with $P \le 4$. In Figure 1, we display this time-frequency for a noiseless quadratic FM signal. Observe that the peaks of the distribution reveal the IF law of the signal. In the presence of complex circular Gaussian multiplicative and additive noises, as in Equation (1), the sixth order PWVD distribution can still reveal the IF of the signal, but its performance is degraded. Figure 2 illustrates this point. In this last figure, we chose $(|\mu_m| = 3, \sigma_m^2 = 1)$ for the Gaussian multiplicative noise and $(0, \sigma_w^2 = 1)$ for the Gaussian additive one.

In the next section, we will consider an arbitrary order polynomial FM signal corrupted by multiplicative and additive complex Gaussian noises and use the peak of the appropriate order PWVD to estimate the signal IF. We, then, evaluate the statistical performance of the estimator. Precisely, we will derive analytical expressions of the bias and variance of such an estimator.

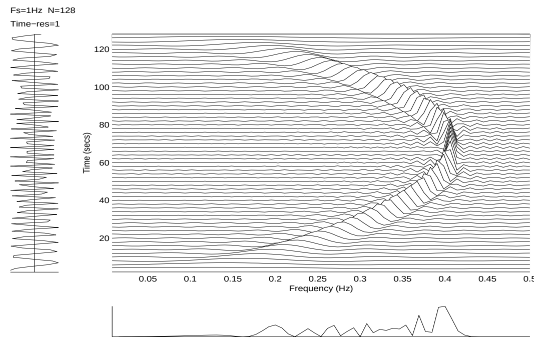

Fig. 1. The sixth order PWVD of a noiseless quadratic FM signal

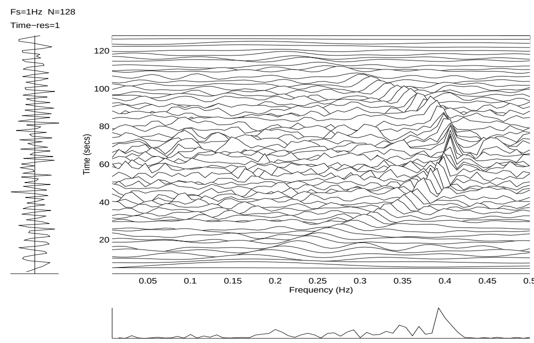

Fig. 2. The sixth order PWVD of a quadratic FM signal corrupted by multiplicative and additive noises.

4. BIAS AND VARIANCE OF THE IF ESTIMATOR

Consider the IF estimation of the signal $z(t)$ in Equation (1). From the results in Section 2, we know that this problem is equivalent to the estimation of the IF of $z_1(t) = |\mu_m| \cdot e^{j(\Theta_{\mu_m}+\phi(t))}$ in Equation (2) (recall that $z(t)$ and $z_1(t)$ have the same IF, as shown in Section 2). We can, thus, state the discrete-time version of the problem as follows.

Consider the discrete-time version of the model in Equation (2), namely,

$$y(nT) = z_1(nT) + w_1(nT), \qquad n = 0, \ldots, N-1. \quad (6)$$

where $z_1(nT)$ is the polynomial FM signal whose IF is to be estimated (and whose amplitude is equal to $|\mu_m|$) and $w_1(nT)$ is a zero-mean circular complex Gaussian noise with variance equal to $\sigma_{w_1}^2 = (\sigma_m^2 + \sigma_w^2)$; N is the number of observations and T the sampling period. Without loss of generality, we will assume $T = 1$. The PWVD (in discrete-time form) of the noisy signal $y(n)$ is given by

$$\begin{aligned} W_y^{(q)}(n,f) &= \sum_{m=-M}^{M} \prod_{i=1}^{q/2} [z_1(n + c_i m) + w_1(n + c_i m)] \\ &\times [z_1^*(n - c_i m) + w_1^*(n - c_i m)]\, e^{-j2\pi fm} \end{aligned} \quad (7)$$

The kernel of the noisy signal expressed as

$$\begin{aligned}K_y^{(q)}(n,m) &= \prod_{i=1}^{q/2}[z_1(n+c_im)+w_1(n+c_im)] \\ &\times \quad [z_1^*(n-c_im)+w_1^*(n-c_im)]\end{aligned}$$

can easily be shown to be equal to [11]

$$K_y^{(q)}(n,m) = K_{z_1}^{(q)}(n,m) + C_T(n,m) + \cdots \tag{8}$$

where

$$K_{z_1}^{(q)}(n,m) = \prod_{i=1}^{q/2}[z_1(n+c_im)z_1^*(n-c_im)]$$

and

$$\begin{aligned}C_T(n,m) &= \left[\prod_{i=1}^{n_1} z_1^{k_i}(n+c_im)z_1^{*(k_i)}(n-c_im)\right] \\ &\times \left\{\sum_{i=1}^{n_1} k_i\left[z_1^{-1}(n+c_im)w_1(n+c_im)\right.\right. \\ &\left.\left. + z_1^{*(-1)}(n-c_im)w_1^*(n-c_im)\right]\right\}\end{aligned}$$

with n_1 being the number of the different coefficients c_i in the kernel, and k_i, $i = 1, \ldots, n_1$, being the multiplicity of each coefficient c_i. Note that $\sum_{i=1}^{n_1} k_i = q/2$, and $c_i \neq c_j \ \forall\ i \neq j$ [10].

The terms indicated by dots, in Equation (8), correspond to quantities which involve products of more than one noise term. If we assume the power in $K_{z_1}^{(q)}(n,m)$ and $C_T(n,m)$ much larger than that in the discarded terms, we can write

$$K_y^{(q)}(n,m) \approx K_{z_1}^{(q)}(n,m) + C_T(n,m)$$

Using the linearity of the expectation operator and the i.i.d property of the complex Gaussian noise $w_1(n)$, we can easily show that the power in $C_T(n,m)$, for a fixed non-zero lag m, is equal to

$$\begin{aligned}P &= \mathrm{E}\left[C_T(n,m)\cdot C_T^*(n,m)\right] \\ &= 2\cdot|\mu_m|^{2(q-1)}\cdot\sigma_{w_1}^2\cdot\sum_{i=1}^{n_1}k_i^2.\end{aligned}$$

The IF estimate, for each time instant, is obtained at the frequency at which the PWVD has its peak. That is, The PWVD based estimate $\hat{f}_i$ of the true IF f_i is defined as the frequency at which the real-valued function $|W_y^{(q)}(n,f)|^2$ is maximum, i.e.,

$$\left[\frac{\partial}{\partial f}|W_y^{(q)}(n,f)|^2\right]_{f=\hat{f}_i} = 0. \tag{9}$$

To determine $\hat{f}_i$, we consider the expansion of $|W_y^{(q)}(n,f)|^2$ up to second order around $(f - f_i)$, i.e.,

$$\begin{aligned}|W_y^{(q)}(n,f)|^2 &\approx |W_y^{(q)}(f_i)|^2 + (f-f_i)\left[\frac{\partial|W_y^{(q)}(n,f)|^2}{\partial f}\right]_{f=f_i} \\ &+ \frac{1}{2}(f-f_i)^2\left[\frac{\partial^2|W_y^{(q)}(n,f)|^2}{\partial f^2}\right]_{f=f_i}\end{aligned} \tag{10}$$

Differentiating (10) with respect to f results in

$$\begin{aligned}\frac{\partial|W_y^{(q)}(n,f)|^2}{\partial f} &\approx \left[\frac{\partial|W_y^{(q)}(n,f)|^2}{\partial f}\right]_{f=f_i} \\ &+ (f-f_i)\left[\frac{\partial^2|W_y^{(q)}(n,f)|^2}{\partial f^2}\right]_{f=f_i}\end{aligned} \tag{11}$$

and using equation (9) we obtain:

$$\hat{f}_i \approx f_i - \left\{\left[\frac{\partial|W_y^{(q)}(n,f)|^2}{\partial f}\right]_{f=f_i} \Big/ \left[\frac{\partial^2|W_y^{(q)}(n,f)|^2}{\partial f^2}\right]_{f=f_i}\right\}. \tag{12}$$

Taking the expected value of (12), the expressions for the bias and variance in the estimate of f_i are found to be

$$\textbf{Bias}\big(\hat{f}_i(n)\big) \approx -\mathrm{E}\left[\frac{\left[\frac{\partial|W_y^{(q)}(n,f)|^2}{\partial f}\right]_{f=f_i}}{\left[\frac{\partial^2|W_y^{(q)}(n,f)|^2}{\partial f^2}\right]_{f=f_i}}\right] \tag{13}$$

$$\textbf{Var}\big(\hat{f}_i(n)\big) \approx \mathrm{E}\left[\left\{\frac{\left[\frac{\partial|W_y^{(q)}(n,f)|^2}{\partial f}\right]_{f=f_i}}{\left[\frac{\partial^2|W_y^{(q)}(n,f)|^2}{\partial f^2}\right]_{f=f_i}}\right\}^2\right] \tag{14}$$

Using similar derivations as those in [11] (Appendix A), the previous expressions result in

$$\textbf{Bias}(\hat{f}_i(n)) = 0 \tag{15}$$

$$\begin{aligned}\textbf{Var}(\hat{f}_i(n)) &\approx \frac{6\ \sigma_{w_1}^2\sum_{i=1}^{n_1}k_i^2}{(2\pi)^2|\mu_m|^2M(M+1)(2M+1)} \\ &\approx \frac{6\ (\sigma_m^2+\sigma_w^2)\sum_{i=1}^{n_1}k_i^2}{(2\pi)^2|\mu_m|^2M(M+1)(2M+1)}\end{aligned} \tag{16}$$

In the last equation, we used the analysis result of Section 2, namely $\sigma_{w_1}^2 = (\sigma_m^2 + \sigma_w^2)$.

Note that the variance decreases with increasing values of M or increasing values of the window length $(2M+1)$ in the PWVD (refer to Equation (7)) . In the middle of the time interval (assuming N odd) where the window length can be chosen equal to the signal length (i.e., $2M+1 = N$), the variance is minimum and equal to

$$\textbf{Var}(\hat{f}_i) = \frac{24\ (\sigma_m^2+\sigma_w^2)\sum_{i=1}^{n_1}k_i^2}{(2\pi)^2|\mu_m|^2N(N^2-1)}. \tag{17}$$

At other time instants n, the implementation of the PWVD necessitates smaller window lengths $(2M+1 < N)$ resulting in a decrease in the statistical performance of the estimator.

For the particular case where $|\mu_m| = A$ (constant) and $\sigma_m = 0$, i.e., the signal under consideration is just a constant amplitude polynomial FM signal embedded in complex Gaussian noise, the IF estimator variance expression becomes

$$\textbf{Var}(\hat{f}_i) = \frac{6\ \sigma_w^2\sum_{i=1}^{n_1}k_i^2}{(2\pi)^2A^2M(M+1)(2M+1)}$$

which is exactly the result obtained in [11], where we treated *constant* amplitude polynomial FM signals only.

5. EXAMPLE

Let us consider the IF estimation of a quadratic FM signal at the middle of the signal interval. The peak of the sixth order PWVD, whose kernel is given by (5), is used here as the IF estimator. The noiseless signal $z(t)$ is modulated by a complex circular Gaussian noise $m(t)$ (μ_m, σ_m^2) and, then, added to a complex circular Gaussian noise $w(t)$ $(0, \sigma_w^2)$, as suggested in Equation (1). The noisy signal $y(t)$ is sampled at $T = 1$ and the number of observations N is chosen equal to 129. The additive and multiplicative noise processes are assumed independent. In the simulations, the overall signal-to-noise ratio (SNR), defined as $\mathrm{SNR}_{w_1} = 10\log_{10}(|\mu_m|^2/(\sigma_m^2 + \sigma_w^2))$, is varied in a 1 dB step from 0 to 15 dB. Monte-Carlo simulations for 1000 realizations are run for each value of SNR_{w_1}. The results of two different experiments, one conducted for $|\mu_m| = 0.01$ and the other conducted for $|\mu_m| = 1$, are displayed in Figure 3. We observe that, above a certain threshold, the estimated variances represented by '+' (for $|\mu_m| = 0.01$) and 'o' (for $|\mu_m| = 1$) are in agreement with the derived theoretical ones given by Equation (16) and represented by the continuous lines (superimposed). In this example, we fixed $\sigma_m^2 = \sigma_w^2$ and $(2M + 1 = N = 129)$. We used other values for $|\mu_m|$ and considered the case $\sigma_m^2 \neq \sigma_w^2$ and found similar results, i.e., the estimated results confirm the theoretical ones. For other polynomial FM signals, the simulations results were in agreement with the theoretical ones (see Figure 4 for the case of a linear FM signal where the peak of the WVD is used as an IF estimator).

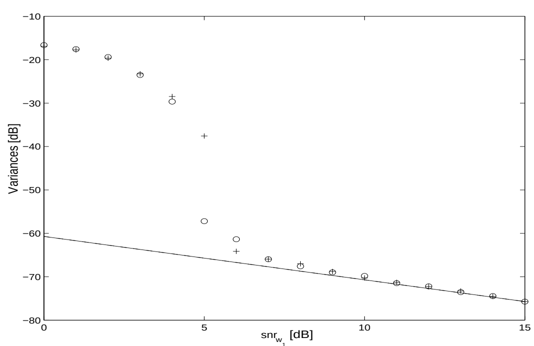

Fig. 3. Results of the experiments for a **quadratic** FM signal corrupted by complex Gaussian multiplicative and additive noise processes. The continuous lines (superimposed) represent the theoretical variances while '+' and 'o' correspond to the estimated variances for $|\mu_m| = 0.01$ and $|\mu_m| = 1$ respectively.

6. CONCLUSION

In this paper, we considered the IF estimation of polynomial FM signals affected by multiplicative and additive complex circular Gaussian noises and proposed the peak of the PWVD as an estimator. We derived analytical expressions of the bias and asymptotic variance of the estimator. We showed that this estimator is unbiased and its statistical performance can deteriorate when using small window lengths in the implementation of the PWVD distribution. We presented some examples, based on Monte-Carlo simulations, to illustrate and validate the theoretical derivations and analysis.

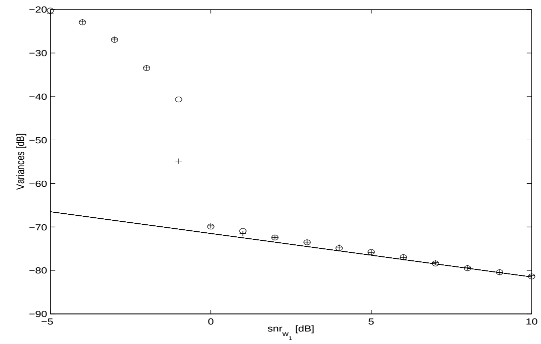

Fig. 4. Results of the experiments for a **linear** FM signal corrupted by complex Gaussian multiplicative and additive noise processes. The continuous lines (superimposed) represent the theoretical variances while '+' and 'o' correspond to the estimated variances for $|\mu_m| = 0.01$ and $|\mu_m| = 1$ respectively.

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