PERFORMANCE ANALYSIS AND COMPARISON OF MRC AND OPTIMAL COMBINING IN ANTENNA ARRAY SYSTEMS

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ABSTRACT

In this paper we examine the statistical properties of the output signal to interference plus noise ratio (SINR) of a spatial combiner where the spatial weights used are either the Maximal Ratio Combiner (MRC) weights or the Optimal Combining (Weiner-Hopf) weights. The channels are modeled as slow flat Rayleigh fading channels and multiple interferers are assumed present. In particular, the modified F-distribution is introduced to provide an exact characterization of a MRC receiver and to bound the performance of the OC receiver. Simulation results are provided to support the analytical results and to provide insight.

1. INTRODUCTION

Adaptive arrays have received a great deal of attention in wireless communication systems. The spatial degree of freedom afforded by antenna arrays can be used to mitigate the adverse effects of multipath and cochannel interference. In this paper, we consider the use of an antenna array in the reverse link and develop analytical results to quantify the effect of using linear combining in an antenna array based receiver. Of particular interest in this analysis is the case where the weights correspond to a Maximal Ratio Combiner (MRC) and an optimal combiner (OC). The analysis is quite complex and several results have been developed in [1, 2, 3, 4]. This paper continues this analytical tradition and provides useful extension of these results. In particular, an exact expression for the density function of the SINR for MRC along with useful bounds for OC in terms of the modified F-distribution are provided.

For the analysis, the received signal at the output of the array of M antennas at time instant n is modeled as

$$Y_n = V_1 d_n^{(1)} + \sum_{k=2}^{N} V_k d_n^{(k)} + N_n.$$

The desired user is indexed by $k = 1$. V_n denotes the spatial signature of user n, and is written as $V_n = \sqrt{P_n}\, c_n$. P_n is the power in the signal and c_n is the fast fading component. It is assumed that c_n is circularly Gaussian denoted as $c_n \sim C(0, I)$, i.e. $E(c_n) = 0$, $E(c_n c_n^H) = I_{M\times M}$, $E(c_n c_n^T) = 0_{M\times M}$. The desired user is assumed to have power P_d, i.e. $P_1 = P_d$, and all the $(N-1)$ interferers are assumed to be of equal power, i.e. $P_n = P_I, n \neq 1$. The desired user power (P_d) is distinguished from the interferers (P_I) to accommodate the spreading gain when considering CDMA systems. N_n is the thermal noise component with $N_n \sim C(0, I)$ and is also temporally white, i.e. $E(N_k N_l^H) = \sigma_N^2 I \delta[k-l]$. Without loss of generality, σ_N^2 is set equal to one. Non unity variance can be readily accomodated by changing P_d and P_I to $\frac{P_d}{\sigma_N^2}$ and $\frac{P_I}{\sigma_N^2}$ in the expressions derived. Hence for the analysis that follows, P_d and P_I can be interpreted as the desired user and interferer signal to thermal noise ratio's respectively. $d_n^{(1)}$ is the information bits of the user that are assumed temporally independent, zero mean, and of variance 1. $d_n^{(k)}$ is related to the information bits in the kth user. It is also assumed to be temporally independent, independent of other users, zero mean, and of variance 1.

The interference plus thermal noise covariance matrix is denoted by R_I, and is given by

$$R_I = \sum_{n=2}^{N} V_n V_n^H + I = P_I \sum_{n=2}^{N} c_n c_n^H + I.$$

Here the averaging is assumed to be over a small enough time interval that the spatial signatures can be considered essentially unchanged. R_I is a random matrix and has to be considered in the statistical analysis.

For the analysis, the key parameter of interest is the Signal to Interference plus Noise Ratio (SINR) at the output of the spatial combiner. It is given by

$$SINR = \frac{|W^H V_1|^2}{W^H R_I W} = \frac{P_d |W^H c_1|^2}{W^H R_I W},$$

where W is the spatially combining weight vector. In Section 2, we analyze MRC combining and in Section 3 the

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optimal combining method. In Section 4, simulation results are presented to provide insight and support the theory.

2. MAXIMAL RATIO COMBINING (MRC)

In Maximal Ratio Combining (MRC), the weight vector W is chosen to maximize the signal to (thermal) noise ratio (SNR). The weight vector is chosen as

$$W_{MRC} = c_1. \quad (1)$$

The SINR in this case is given by

$$SINR_{MRC} = P_d \frac{(c_1^H c_1)^2}{c_1^H R_I c_1}. \quad (2)$$

First, we present an approximate analysis because of the simplicity and intuitive appeal. Then an exact analysis is provided.

2.1. Approximate Analysis

For this analysis, we assume a large number of users, a scenario consistent with a CDMA system with a large number of low power users. In the case where there are a large number of users, the law of large numbers can be employed to simplify the analysis. Using the law of large numbers, we have

$$\frac{1}{N-1} R_I \xrightarrow{a.s.} P_I I + \frac{1}{N-1} I = \sigma^2 I, \quad (3)$$

where $\sigma^2 = (P_I + \frac{1}{N-1})I = \frac{1}{N-1}((N-1)P_I + 1)I$. Using (3), leads to the following simple expression for the SINR

$$SINR_{MRC} = \frac{P_d}{(N-1)\sigma^2} c_1^H c_1 = \frac{P_d}{(N-1)\sigma^2} Z. \quad (4)$$

where $Z = c_1^H c_1$.

$$Z = c_1^H c_1 = \sum_{k=1}^{M} |c_k^{1,r}|^2 + \sum_{k=1}^{M} |c_k^{1,i}|^2,$$

where the elements of the vector c_1 are the complex entries c_k^1 with real and imaginary components $c_k^{1,r}$ and $c_k^{1,i}$ respectively, i.e. $c_k^1 = c_k^{1,r} + jc_k^{1,i}$. Because $c_1 \sim C(0, I)$, $2Z$ is a central chi-squared random variable with $2M$ degrees of freedom [5]. This is denoted by χ^2_{2M} and an explicit expression for the density function of Z is given by

$$f_Z(M, z) = \frac{1}{\Gamma(M)} z^{M-1} e^{-z} u(z), \quad (5)$$

where $\Gamma(.)$ is the Gamma function and $u(z)$ is the unit step function. $E(Z) = M$, and $\mathrm{Var}(Z) = \sigma_Z^2 = M$. The cumulative distribution function is given by [5]

$$F_Z(M, z) = (1 - e^{-z} \sum_{k=0}^{M-1} \frac{1}{k!} z^k) u(z). \quad (6)$$

Defining $S = SINR_{MRC} = \frac{P_d}{(N-1)P_I+1} Z$, it follows that

$$\begin{aligned} F_S(s) &= F_Z(M, \frac{(N-1)P_I+1}{P_d} s) \text{ and} \\ f_S(s) &= \frac{(N-1)P_I+1}{P_d} f_Z(M, \frac{(N-1)P_I+1}{P_d} s) \quad (7) \end{aligned}$$

Remarks:

1. $F_Z(M, z)$ is dependent only on the number of antennas, a factor that simplifies outage probability and capacity computation.
2. $F_S(s)$ is obtained from $F_Z(z)$ by a simple scaling of the variable.

A more exact analysis can be conducted carrying the randomness of R_I to the end. However, the approximate analysis may have more general applicability because the law of large numbers may be employed under less stringent conditions. For instance, it is possible to extend the analysis to the more general Nakagami fading case.

2.2. Exact Analysis

A more exact analysis is possible as discussed below and the results are valid irrespective of the relationship between N, the number of users, and M, the number of antennas. Defining $\tilde{c}_1 = \frac{c_1}{\|c_1\|}$,

$$\begin{aligned} SINR_{MRC} &= P_d \frac{c_1^H c_1}{\tilde{c}_1^H R_I \tilde{c}_1} = \frac{P_d}{P_I} \frac{c_1^H c_1}{\tilde{c}_1^H X X^H \tilde{c}_1 + b} \\ &= \frac{P_d}{P_I} \frac{\chi^2_{2M}}{2b + \chi^2_{2(N-1)}} = \frac{P_d}{P_I} F \quad (8) \end{aligned}$$

Note that the SINR is ratio of two independent random variables with the numerator being a χ^2_{2M} random variable and the denominator being a translated $\chi^2_{2(N-1)}$ random variable. For this purpose, an useful extension of the F-distribution termed the modified-F distribution is relevant. It is defined as the distribution of the random variable

$$F = \frac{X}{Y + b}$$

where $2X$ is χ^2_{2p}, $2Y$ is χ^2_{2q}, and X and Y are independent r.v's. b is a deterministic constant. The density function of F can be shown to be [7]

$$\begin{aligned} f_{M-F}(p, q, b, f) = & \frac{1}{\Gamma(p)\Gamma(q)} f^{p-1} e^{-bf} \\ & \sum_{l=0}^{p} \binom{p}{l} b^{p-l} \frac{\Gamma(l+q)}{(1+f)^{l+q}} u(f). \quad (9) \end{aligned}$$

The corresponding distribution is denoted by $F(p, q, b, f)$. From (8), the density function of the SINR is given by [7]

$$f_S(s) = \frac{P_I}{P_d} f_{M-F}(M, N-1, b, \frac{P_I}{P_d} f), \quad (10)$$

If b can be neglected, the density function has a simple form and is related to the F-distribution. This is true if we are interference limited such as in the case of high powered users. Then only one term from the summation in (10) is non-zero and we have

$$f_S(s) = \frac{P_I}{P_d}\, f_F(M, N-1, \frac{P_I}{P_d} f), \tag{11}$$

where

$$f_F(p,q,f) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \frac{f^{(p-1)}}{(1+f)^{(p+q)}} u(f). \tag{12}$$

Note that $f_S(.)$ only depends on $\frac{P_d}{P_I}$, the desired user to interferer power.

3. OPTIMAL COMBINING

The Optimum Combining (OC) weights are chosen to maximize the SINR and the weight vector is given by

$$W_{OC} = R_I^{-1} c_1. \tag{13}$$

The SINR is given by

$$SINR_{OC} = P_d\, c_1^H R_I^{-1} c_1. \tag{14}$$

The OC weights involve more complex computation and reduce to the Maximal Ratio Combiner (MRC) weights (1) when the interference matrix is spatially white. In the presence of interferers,

$$R_I = P_I \sum_{n=2}^{N} c_n c_n^H + I = P_I\, XX^H + I,$$

where $X = [c_2, .., c_N]$. Rearranging, the terms we have

$$SINR_{OC} = \frac{P_d}{P_I}\, c_1^H (\frac{1}{P_I} I + XX^H)^{-1} c_1 = \frac{P_d}{P_I}\, S_t, \tag{15}$$

where $S_t = c_1^H (bI + XX^H)^{-1} c_1$, and $b = \frac{1}{P_I}$. For the analysis, it is useful to consider two different scenarios: Overloaded and the Underloaded case.

3.1. Overloaded Case

In this section, we assume that there are more users than antennas, i.e. $(N > M)$ or $(N-1) \geq M$.

3.1.1. Large number of Users

Because of the large number of users, as done in the MRC case (c.f. section 2.1), the law of large numbers can be made use of (3) [8]. Using (3) in (14), we have

$$SINR_{OC} = \frac{P_d}{(N-1)\sigma^2} c_1^H c_1 = \frac{P_d}{(N-1)P_I + 1} \frac{1}{2} \chi^2_{2M}. \tag{16}$$

This is the same as the SINR expression for MRC (c.f. (4) in Section 2.1) . Consequently, for this special case there is no difference between OC and the simpler MRC method.

3.1.2. High Power Users

Here we assume that the users are of high power and that the thermal noise can be neglected. Then P_I is large and $b \to 0$. Then

$$SINR_{OC} = \frac{P_d}{P_I}\, c_1^H (XX^H)^{-1} c_1.$$

This approximate form has a nice closed distribution function [2]. Also noteworthy is the dependence on the ratio of the users to interferer power. If $G = c_1(XX^H)^{-1}c_1$, XX^H has a Wishart distribution $W_{N-1}(0, I)$, and so the density function of G is given by [2, 6]

$$f_S(s) = \frac{P_I}{P_d}\, f_F(M, N-M, \frac{P_I}{P_d} s). \tag{17}$$

To compare the result with that obtained for MRC, the following alternate form of $SINR_{OC}$ is useful [7]

$$SINR_{OC} = \frac{P_d}{P_I}\, \frac{\chi^2_{2M}}{\chi_{2(N-M)}}. \tag{18}$$

Comparing (18) with (8), it can be observed that the OC behavior has the same form as MRC with $(M-1)$ fewer users.

3.1.3. Useful Bounds

More general results that do not make the assumptions of Sections 3.1.1 and 3.1.2 can be found in [4]. One difficulty with the results is their tractability. Here we provide some useful bounds that are more tractable and can help in the understanding. It can be shown that [7]

$$\frac{P_d}{P_I}\, \frac{\chi^2_{2M}}{2b + \chi^2_{2(N-1)}} \leq SINR_{OC} \leq \frac{P_d}{P_I}\, \frac{\chi^2_{2M}}{2b + \chi^2_{2(N-M)}},$$

and hence

$$F_{M-F}(M, N-1, b, \frac{P_I}{P_d} s) \leq F_S(s) \leq F_{M-F}(M, N-M, b, \frac{P_I}{P_d} s).$$

Note that the lower bound corresponds to MRC.

3.2. Underloaded Case

We assume that the number of users N is smaller than or equal to the number of antenna M, i.e. $N \leq M$, or the number of interferers ($(N-1)$) are strictly less than M. It can be shown that [7]

$$SINR_{OC} = S = S_1 + S_2.$$

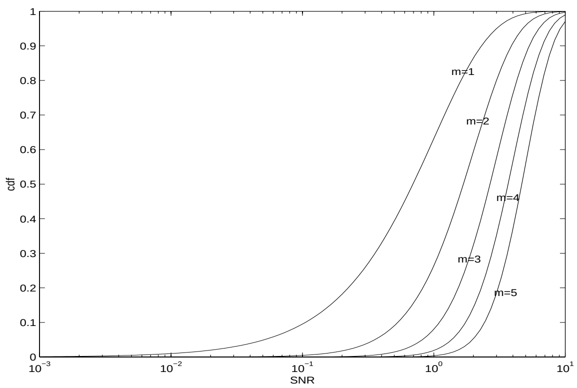

Fig. 1. Many low data rate users. The Chi-Squared Distribution as a function of the number of Antennas

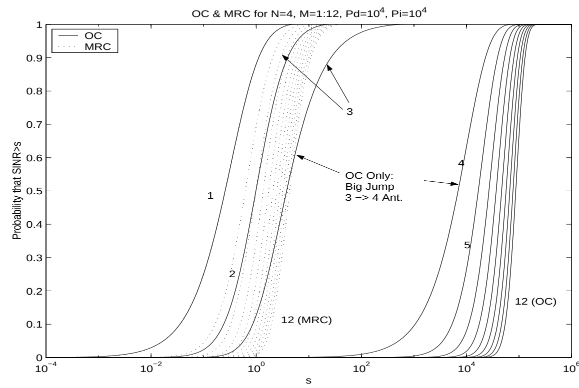

Fig. 2. Performance of $N = 4$ high power users as a function of number of antennas ranging from 1 to 12.

where S_1 and S_2 are independent r.v.'s. Hence

$$f_S(s) = f_{S_1}(s) * f_{S_2}(s), \tag{19}$$

where $*$ denotes the convolution operation.

$$S_2 = P_d\ c_1^H P_{\mathcal{R}(X)}^{\perp} c_1 = P_d\ \frac{1}{2}\chi^2_{2(M-N+1)}.$$

and the density function of S_2 is given by

$$f_{S_2}(s) = \frac{1}{P_d} f_Z(M - N + 1, \frac{1}{P_d}s). \tag{20}$$

S_1 can be shown to have the same distribution as the random variable [7]

$$P = \frac{P_d}{P_I}\ V^H(bI_{N-1} + X^H X)^{-1} V$$

where V is complex $C(0, I)$ circularly Gaussian random vector with $(N - 1)$ components. For high power users,

$$f_{S_1}(s) = \frac{P_I}{P_d}\ f_F(N - 1, M - N + 2, \frac{P_I}{P_d}s). \tag{21}$$

Using (21) and (20) in (19), one can compute the density of a small number of high power users. More results on the density function of S_1 can be found in [7].

4. SIMULATION RESULTS

Monte Carlo simulations have been performed to validate the theory. Due to the excellent agreement between the theory and simulations, we only provide the theoretical plots and try to contrast and provide some insight into the methods.

Example 1: We first consider the case of many low data users. The MRC and OC have similar performance and the SINR distribution is defined by the Chi-Squared distribution (c.f. (6)). Plots of the distribution $F_Z(M, z)$ as a function of number of antennas M are shown in figure 1. The distribution of the SINR can be obtained by translating these plots.

Example 2: In this example there are four high data users. The distribution function for the SINR are plotted as a function of the number of antennas in Figure 2. As evident from the figure, OC performs better than MRC. In the underloaded case, more antennas than users, the superior interference suppression capability of OC is evident. Even in the overloaded case, OC has superior performance. This can be gleaned from the density funtion of the SINR for OC where the number of antennas plays a role in both the variables (p, q) of the F-distribution (c.f. (17)) as opposed to only the p variable in the MRC approach (c.f. (11)).

5. REFERENCES

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