ANALYSIS OF NON-STATIONARY MODE COUPLING BY MEANS OF WAVELET-BICOHERENCE

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ABSTRACT

We present a definition of wavelet-bicoherence based on wavelet-polyspectra. We propose a simple estimator for wavelet-bicoherence, and discuss its statistical properties. In particular it is shown that wavelet-bicoherence estimates has a larger number of effective degrees of freedom than traditional Fourier-based bicoherence estimates. The proposed estimator is applied to detection of coherent couplings in rocket measurements from the ionospheric E-region. It is concluded that wavelet-bicoherence is a well suited tool for analysis of non-stationary mode coupling.

1. INTRODUCTION

The problem considered in this paper is that of detecting coherent mode couplings of electrostatic waves in the ionospheric E-region from rocket measurements. Signals containing coherent couplings have traditionally been analyzed by means of a normalized bispectrum, called the bicoherence [1]. The bicoherence, when defined properly, quantifies the fraction of power contained in the non-linearity. However, since our available data are obtained by means of a rocket launch, they have a highly non-stationary nature. Thus, traditional FFT-based methods [2, 3] may be inadequate due to the inability of the short-time Fourier transform to resolve short lived transients properly. A recently proposed method which addresses this problem is the wavelet-bicoherence [4, 5, 6], which is based on the continuous wavelet transform. For analysis of non-stationary processes the wavelet-bispectrum has two main advantages compared to traditional FFT-based methods. First, since the continuous wavelet transform is a time-scale representation of a signal, we introduce a time axis in a natural way. Second, wavelets has an inherent constant-Q filtering property, and are consequently well suited for detection of transients.

This paper is organized as follows. In section 2 we define the wavelet-bicoherence in terms of the wavelet-polyspectra introduced by the authors in [6]. Then we propose an estimator for the wavelet-bicoherence, and discuss the statistical properties of the resulting estimate. In section 3, to demonstrate the method's performance under non-stationary conditions, we apply the proposed estimator to measurements of electrical potentials in the ionosphere.

2. WAVELET-BICOHERENCE

2.1. Definitions

The continuous wavelet transform (CWT) of a real valued signal $x(t)$ is defined as [7]

$$W_\psi(a,t) \triangleq \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t') \psi^* \left(\frac{t'-t}{a} \right) dt'. \quad (1)$$

Here, $\psi(t)$ is an admissible wavelet, a is a scale variable and t is a time variable. Then the nth-order wavelet-polyspectrum with respect to an admissible wavelet $\psi(t)$ is defined as [6]

$$M_n^w(a_1, \dots, a_{n-1}, t) \triangleq$$
$$\frac{1}{T} E \left\{ \int_{t-T/2}^{t+T/2} W_\psi(a_1, t') \cdots W_\psi(a_{n-1}, t') W_\psi^*(a, t') \, dt' \right\}, \quad (2)$$

where the process $X(t)$ is assumed to be stationary on the local interval of integration $[t - T/2, t + T/2]$ and

$$a^{-1} = a_1^{-1} + a_2^{-1} + \dots + a_{n-1}^{-1}. \quad (3)$$

The wavelets used in this paper are of the form

$$\psi(t) = g(t) \exp(j\eta t), \quad (4)$$

where $g(t)$ is a real valued and symmetric window which has to be well localized in the frequency domain. The parameter η is chosen such that the Fourier transform of $\psi(t)$ is essentially zero for $f \leq 0$. With this choice of wavelet there is a well defined relationship between instantaneous frequency f and scale a given by $a = \eta/(2\pi f)$ for $f \neq 0$ [7]. Note that the CWT is not defined for frequency $f = 0$.

For zero-mean processes we define the wavelet-bispectrum by $B_w(f_1, f_2, t) \triangleq M_3^w(f_1, f_2, t)$ and the wavelet power spectrum by $S_w(f, t) \triangleq M_2^w(f, t)$, where we have replaced the scales a with $\eta/(2\pi f)$ as explained in the previous paragraph. A normalized version of the wavelet-bispectrum is then the squared wavelet-bicoherence defined by

$$b_w^2(f_1, f_2, t) \triangleq$$
$$\frac{|B_w(f_1, f_2, t)|^2}{\frac{1}{T} E \left\{ \int_{t-T/2}^{t+T/2} |W_\psi(f_1, t') W_\psi(f_2, t')|^2 \, dt' \right\} \cdot S_w(f_1 + f_2, t)}, \quad (5)$$

where again the process analyzed is assumed to be stationary on the local interval $[t - T/2, t + T/2]$. This quantity may easily be shown to be bounded by $0 \leq b_w^2(f_1, f_2, t) \leq 1$ by employing Cauchy-Schwarz' inequality. The quantity $b^2(f_1, f_2, t)$ is a measure of the fraction of power at frequency $f_1 + f_2$ due to the quadratic coupling of the modes at frequencies f_1 and f_2, as a function of time.

2.2. Estimates of the wavelet-bicoherence

Estimation of $b_w^2(f_1, f_2, t)$ is done in a straightforward manner. First we have to discretize the CWT given in Eq. (1). This yields

$$\widehat{W_\psi}(f, k) = \sqrt{\frac{2\pi f}{\eta}} \sum_{m=0}^{N-1} x(m) \psi^* \left[\frac{2\pi f}{\eta}(m - k) \right], \quad (6)$$

assuming sampling time $\Delta t = 1$. Then we estimate the wavelet power spectrum and the wavelet-bispectrum by

$$\widehat{S_w}(f, n) = \frac{1}{M} \sum_{k=n-\lceil M/2 \rceil}^{n+\lceil M/2 \rceil - 1} \left| \widehat{W_\psi}(f_1, k) \right|^2 \quad (7)$$

and

$$\widehat{B_w}(f_1, f_2, n) = \frac{1}{M} \sum_{k=n-\lceil M/2 \rceil}^{n+\lceil M/2 \rceil - 1} \widehat{W_\psi}(f_1, k) \widehat{W_\psi}(f_2, k) \widehat{W_\psi}^*(f_1 + f_2, k), \quad (8)$$

respectively. Here, n is the middle sample, $\lceil \cdot \rceil$ is the ceiling function, and M the number of samples in the local interval of integration, respectively. Substituting the estimates given in Eqs. (6), (7), and (8) into Eq. (5) we get an estimate $\widehat{b_w^2}(f_1, f_2, n)$ of the squared wavelet-bicoherence.

Small values in the denominator estimate may cause ill-conditioned wavelet-bicoherence estimates. This effect is mitigated by adding a small constant C to the denominator, a process which is often referred to as regularization. A disadvantage of this method is that C introduces a negative bias to the overall estimate. Thus, we must select C large enough to remove spurious effects due to small values in the denominator, but no so large as to cause significant bias in the overall estimate.

2.3. Statistical properties

The estimates of $S_w(f, t)$ and $B_w(f_1, f_2, t)$ are, up to frequency dependent factors, asymptotically unbiased estimates of slowly varying power- and bispectra in the limit when M, the number of samples within the (fixed) local interval of integration, tends to infinity, ie. $\Delta t \to 0$ [6]. However, this does not guarantee that $\widehat{b_w^2}$ will be unbiased. Nevertheless, it is often a good approximation to assume asymptotic unbiasedness.

It is not easy to obtain a useful expression for the statistical variability of the wavelet-bicoherence estimator. Commonly one neglects the variability of the denominator, since the variance of the bispectrum estimate in the numerator usually is significantly larger than the variance of the power spectrum estimates in the denominator [1]. Thus, a crude approximation of the estimator
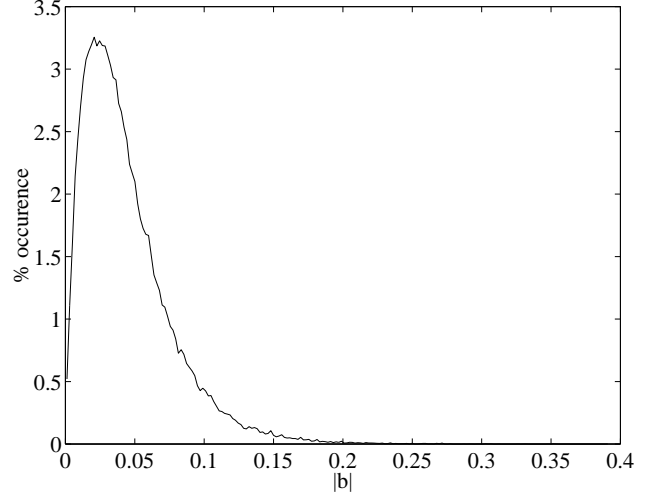
Fig. 1. Observed distribution of $\sqrt{\widehat{b_w^2}}$ estimated from 100 independent realizations of a white Gaussian process, each consisting of 100 samples.

variance is given by

$$\mathrm{Var}\left\{ \widehat{b_w^2}(f_1, f_2, n) \right\} \simeq$$
$$\frac{\mathrm{Var}\left\{ \widehat{B_w}(f_1, f_2, n) \right\}}{E\left\{ \widehat{S_w}(f_1, n) \right\} E\left\{ \widehat{S_w}(f_2, n) \right\} E\left\{ \widehat{S_w}(f_1 + f_2, n) \right\}}. \quad (9)$$

The variance of the estimator $\widehat{B_w}(f_1, f_2, n)$ is given in [6], where it is shown to have statistical properties analogous to the weighted overlapped segment averaging method.

The true bicoherence of a Gaussian process is zero. One can show that for zero bicoherence the estimator $\widehat{b_w^2}$ is approximately χ^2 distributed, and thus will have a positive bias. However, the bias can be shown to be negligible. In Fig. 1 we show the observed distribution of $\sqrt{\widehat{b_w^2}}$ estimated from 100 independent realizations of a Gaussian process, each consisting of 100 Gaussian samples. We have shown the distribution of the square root of the estimate to emphasize the χ^2-like shape.

The expressions for the bias and variance FFT-based bicoherence estimators in the general case are given in [8] apply equally well to the wavelet-bicoherence estimator treated in this article if we include a frequency dependent improvement factor [6]. Thus, the wavelet-based estimator has a larger number of effective degrees of freedom. For the Morlet-Grossmann wavelet described in the next section, one may show that the improvement factor for variance is given by

$$\nu(f_1, f_2) \simeq k \frac{f_N}{M} \cdot \frac{f_1^2 + f_2^2 + (f_1 + f_2)^2}{f_1 f_2 (f_1 + f_2)} \quad (10)$$

where k is a constant depending on the choice of parameters in the wavelet, M is the number of points in the local interval of integration, and f_N is the Nyquist frequency. In [9] this factor is empirically found to be $\nu(f_1, f_2) \sim f_N/(M\sqrt{f_1 f_2})$, which is close to the theoretical value given here.
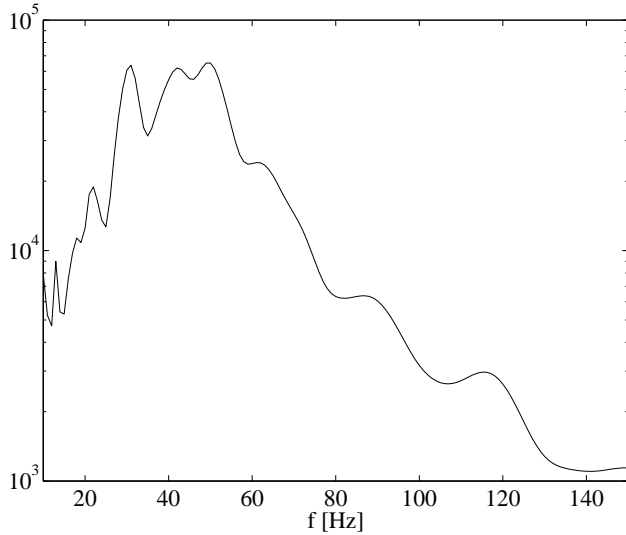
Fig. 2. Estimate of time averaged wavelet power spectrum.

Fig. 3. Estimate of time averaged squared wavelet-bicoherence.

3. DATA ANALYSIS

The measurements of the electrical potentials were obtained by means of a pair of probes mounted on a sounding rocket. The sampling frequency was 2000 Hz. A detailed description of the experimental conditions is found in [10]. Owing to the rocket spin relative to the DC electric field, the electric field signal has a large-amplitude variation following the rocket spin, with the fluctuating wave component being superimposed. Since the spin frequency is far below the frequency range of the phenomena we want to analyze, we remove the fundamental spin frequency and its first few harmonics by applying an 8-Hz zero-phase high-pass filter prior to analysis.

In this paper we will focus on the part of the measured signal obtained between 256 and 258 seconds after launch, which is a 4000 samples long record. This corresponds to the rocket's downleg from approximately 99 to 97 km above the sea level. The signal we analyze is the $u_6(t)$ record [10].

We have in the following analysis used the Morlet-Grossmann wavelet

$$\psi(t) = \exp\left(-\frac{t^2}{2\sigma^2} + j\eta t\right) \qquad (11)$$

as analyzing wavelet. This wavelet is well known for its optimal simultaneous time and frequency resolution, and it thus provides a good general purpose choice. The parameters σ^2 and η are user specified, subject to certain restrictions connected to tradeoff between frequency and time resolution [7, 6]. In this analysis we have chosen $\sigma^2 = 1.5$ and $\eta = 4\pi$.

As an initial analysis, we estimate $S_w(f)$ and $b_w^2(f)$ assuming the time series to be stationary.

The results are shown in Figs. 2 and 3. Note that $\widehat{S_w}(f)$ is multiplied by a constant, which depends on the wavelet parameter σ^2, to yield an approximately unbiased spectrum estimate [6], and that we have restricted the analysis to the low frequency range where we expect to observe mode couplings. The Nyquist frequency is $f_N = 1000$ Hz.
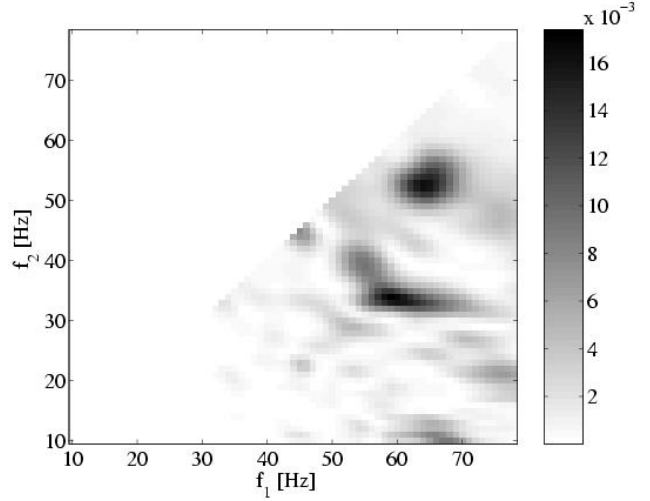
In Fig. 2 we observe several peaks in the frequency range 20-60 Hz. We expect to find couplings between these frequency components in the wavelet-bicoherence estimate. However, as seen in Fig. 3, we find only weak traces of coherent couplings in this analysis. Notice the low coherence level ($\sim 10^{-2}$). This is of course due to the fact that the signal is non-stationary, which causes our estimation procedure to average out the short lived couplings.

Now we repeat the analysis, but this time we assume the process to be approximately stationary only on intervals of 10 samples. The results are shown in Figs. 4 and 5. Again we have restricted the analysis to the low frequency range. In Fig. 4 we can recognize the frequency components observed in Fig. 2, but we clearly see that the process has a non-stationary nature. Fig. 5 shows an isosurface of $\widehat{b_w^2}(f_1, f_2, n)$ at the value 0.77. Again we observe time variable structures, mainly concentrated at frequency pairs corresponding to the frequency components observed in the estimated power spectrum. The value 0.77 is well above the 95% significance level for $b^2 = 0.5$, even for standard bicoherence estimates with the same number of degrees of freedom [8]. Thus, the coherent mode couplings we observe clearly are statistically significant.

4. CONCLUSIONS

We have proposed a definition of wavelet-bicoherence based on wavelet-polyspectra. Furthermore, we have suggested an estimator for wavelet-bicoherence and compared its statistical properties of with those of standard Fourier-based estimators. The estimator is shown to have a larger number of effective degrees of freedom. In particular we have shown that the improvement factor is frequency dependent, due to the constant-Q filtering property of wavelets, and we have provided a theoretical expression for this factor for the Morlet-Grossmann wavelet used in this paper. Next, the wavelet-bicoherence estimator was applied to the analysis and interpretation of non-stationary data observed in the ionospheric E-region. The results show that statistically significant time-varying coherent mode couplings are present at frequencies corresponding to components observed in an initial time averaged power spectral
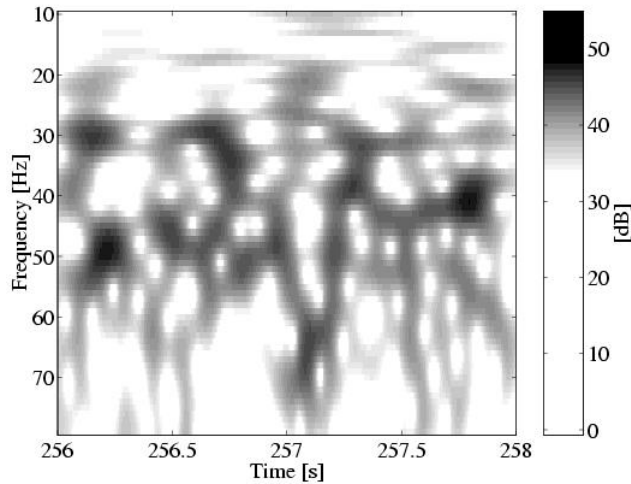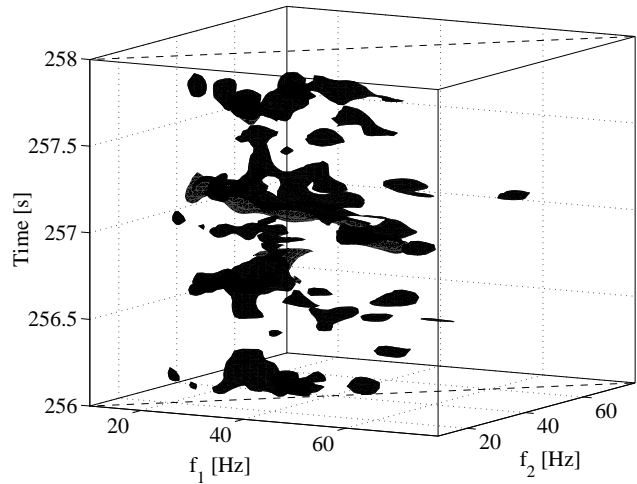
Fig. 4. Estimated evolutionary wavelet power spectrum.

Fig. 5. Estimated evolutionary squared wavelet-bicoherence. Isosurface at $\widehat{b_w^2}(f_1, f_2, n) = 0.77$.

estimate. Thus, the wavelet-bicoherence technique demonstrate a remarkably robust performance, even under highly non-stationary conditions.

5. REFERENCES

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