FAST ALGORITHM FOR LEAST SQUARES 2D LINEAR-PHASE FIR FILTER DESIGN

N.C. Griswold and J. Dávila

Department of Electrical Engineering, DSP Laboratory
Texas A&M University, College Station, TX 77843-3128
Email: griswold@ee.tamu.edu and jdav@ee.tamu.edu

ABSTRACT

In this paper, we develop a new method for weighted least squares 2D linear-phase FIR filter design. It poses the problem of filter design as the problem of projecting the desired frequency response onto the subspace spanned by an appropriate orthonormal basis. We show how to compute the orthonormal basis efficiently in the cases of quadrantally-symmetric filter design and centro-symmetric filter design. The design examples show that the proposed method is faster than a conventional weighted least squares filter design me-thod. Also, the amount of storage required to compute the filter coefficients is greatly reduced.

1. INTRODUCTION

The least-squares design of two-dimensional filters has been studied in a number of references [2]-[5]. This kind of filter design has been studied in [2] for quadrantally-symmetric filters and in [3] and for centro-symmetric filters. Although these methods are extremely fast, they require the specification of a desired frequency response over the whole frequency grid (including the transition band), and do not allow for different weights to be specified on the passband and the stopband. On the other hand, the general method of weighted least squares filter design is very simple and has been well-established [6]. In theory, this method could be applied to the design of weighted least squares filters in two dimensions. However, in practice, the computational complexity can be overwhelming for the two-dimensional case. Moreover, as the filter order increases numerical problems may arise.

In this paper, the aim is to extend the method in [1] to the two dimensional case, both for quadrantally-symmetric filter design and for centro-symmetric filter design. By using this technique, the computational complexity of the general weighted least squares design method is greatly reduced.

2. PRELIMINARY DEFINITIONS

The frequency response of a $(2M+1)\times(2N+1)$ 2D FIR filter is given by

$$H(\phi,\psi)=\sum_{n=-N}^{N}\sum_{m=-M}^{M}h(m,n)e^{-jm\phi}e^{-jn\psi}. \tag{1}$$

In order to obtain a zero-phase filter the following symmetry is usually assumed

$$h(m,n)=h(-m,-n). \tag{2}$$

The filters obtained in this way are known as centro-symmetric filters. When the symmetry of (2) holds, the frequency response of (1) can be written as follows

$$H(\phi,\psi)=\left[\sum_{n=-N}^{N}\sum_{m=1}^{M}2h(m,n)\cos(m\phi+n\psi)\right] + h(0,0)+\sum_{n=1}^{N}2h(0,n)\cos(n\psi). \tag{3}$$

This last equation can also be written as

$$H(\phi,\psi)=\sum_{n=0}^{N}\sum_{m=0}^{N}a_{mn}\cos(m\phi)\cos(n\psi)+ +\sum_{n=1}^{N}\sum_{m=1}^{M}b_{mn}\sin(m\phi)\sin(n\psi). \tag{4}$$

The basic idea of the method proposed in this paper is to project the desired frequency response onto the subspace spanned by an appropriate basis. From (4), the basis of the filter subspace for the centro-symmetric case is $\{\{\{\cos(m\phi)\cos(n\psi)\}_{m=0}^{M}\}_{n=0}^{N}$, $\{\{\sin(m\phi)\sin(n\psi)\}_{m=1}^{M}\}_{n=1}^{N}\}$.

Similarly, the frequency response for a quadrantally-symmetric filter can be expressed as

$$H(\phi,\psi)=\sum_{n=0}^{N}\sum_{m=0}^{M}a_{mn}\cos(m\phi)\cos(n\psi). \tag{5}$$

From (5), the basis of the filter subspace in this case is given by $\{\{\{\cos(m\phi)\cos(n\psi)\}_{m=0}^{M}\}_{n=0}^{N}\}$.

The dot product is defined in the following way

$$\langle f(\phi,\psi), g(\phi,\psi)\rangle = \frac{1}{L_\phi L_\psi}\sum_{i=0}^{L_\phi-1}\sum_{j=0}^{L_\psi-1} W(\phi_i,\psi_j) f(\phi_i,\psi_j) g(\phi_i,\psi_j) \quad (6)$$

where L_ϕ is the number of frequency points along the ϕ direction and L_ψ is the number of frequency points along the ψ direction. The optimal filter coefficients according to the weighted least squares criterion are obtained by projecting the desired frequency response onto the subspace spanned by the appropriate basis and then taking an inverse FFT. Normalized functions are denoted using a bar as in $\bar{f}$.

3. QUADRANTALLY-SYMMETRIC FILTERS

Using Chebyshev polynomials, we can write

$$\cos(m\phi)\cos(n\psi) = (\sum_{k=0}^{m} a_k \cos^k \phi)(\sum_{k=0}^{n} b_k \cos^k \psi). \quad (7)$$

Having this in mind, we now present the following procedure to compute the orthonormal basis for the quadrantally-symmetric case.

Algorithm 1 *Consider the set of functions* $\{\{\bar{f}_{mn}(\phi,\psi)\}_{m=0}^{M}\}_{n=0}^{N}$, *defined in the following recursive way:*

$$f_{0,0}(\phi,\psi) = 1. \quad (8)$$

$$f_{1,0}(\phi,\psi) = \cos(\phi) - \mu_{100}^{(f)}\bar{f}_{0,0}(\phi,\psi). \quad (9)$$

For $m = 2,\ldots,M$,

$$f_{m,0}(\phi,\psi) = \cos(\phi)\bar{f}_{m-1,0} - \sum_{i=m-2}^{m-1}\mu_{m0i}^{(f)}\bar{f}_{i,0}. \quad (10)$$

For $m = 0,\ldots,M$,

$$f_{m,1}(\phi,\psi) = (\cos\psi)\bar{f}_{m,0}(\phi,\psi) - \sum_{i=0}^{M}\alpha_{mni}^{(f)}\bar{f}_{i,0} - \sum_{i=0}^{m-1}\mu_{mni}^{(f)}\bar{f}_{i,n}. \quad (11)$$

For $n = 2,\ldots,N$,

For $m = 0,\ldots,M$,

$$f_{m,n}(\phi,\psi) = \cos(\psi)\bar{f}_{m,n-1} - \sum_{i=0}^{M}\alpha_{mni}^{(f)}\bar{f}_{i,n-1} - \sum_{i=m}^{M}\beta_{mni}^{(f)}\bar{f}_{i,n-2} - \sum_{i=0}^{m-1}\mu_{mni}^{(f)}\bar{f}_{i,n} \quad (12)$$

end.

end.

The coefficients α, β and μ in the previous equations are the dot-products between the first term in the right hand side of the equation and the factor at the right of the coefficient. For example, in (12): $\mu_{mni}^{(f)} = \langle\cos(\psi)\bar{f}_{m,n-1}, \bar{f}_{i,n}\rangle$ for $i = 0,\ldots,m-1$. Care should be taken in implementing the previous algorithm in order to avoid computing the same dot-product more that once.

Theorem 1 The set of functions $\{\{\{\bar{f}_{mn}(\phi,\psi)\}_{m=0}^{M}\}_{n=0}^{N}\}$ computed using Algorithm 1 is an orthonormal basis that spans the same space that the set of functions $\{\{\{\cos(m\phi)\cos(n\psi)\}_{m=0}^{M}\}_{n=0}^{N}\}$ does.

The proof of this theorem can be found in [7].

Algorithm 1 allows us to compute the orthonormal basis efficiently. Once the orthonormal basis has been determined, the actual frequency response can be obtained from

$$H(\phi,\psi) = \sum_{m=0}^{M}\sum_{n=0}^{N}\langle D(\phi,\psi), \bar{f}_{mn}(\phi,\psi)\rangle \bar{f}_{mn}(\phi,\psi). \quad (13)$$

Once the frequency response has been computed, the filter coefficients can be recovered efficiently using an inverse 2D FFT.

4. CENTRO-SYMMETRIC FILTERS

Using again Chebyshev polynomials, $\sin(m\phi)\sin(n\psi)$ can be written as

$$\sin(m\phi)\sin(n\psi) = \sin\phi(\sum_{k=0}^{m-1} c_k\cos^k\phi)\cdot \sin\psi(\sum_{k=0}^{m-1} d_k\cos^k\psi). \quad (14)$$

We now present the following procedure to compute the orthonormal basis for the centro-symmetric case.

Algorithm 2 *Consider the sets of functions* $\{\{\bar{f}_{mn}(\phi,\psi)\}_{m=0}^{M}\}_{n=0}^{N}$, $\{\{\bar{g}_{mn}(\phi,\psi)\}_{m=1}^{M}\}_{n=1}^{N}$, *defined in the following recursive way:*

$$f_{0,0}(\phi,\psi) = 1. \quad (15)$$

$$f_{1,0}(\phi,\psi) = \cos(\phi) - \mu_{100}^{(f)}\bar{f}_{0,0}(\phi,\psi). \quad (16)$$

For $m = 2,\ldots,M$

$$f_{m,0}(\phi,\psi) = \cos(\phi)\bar{f}_{m-1,0} - \sum_{i=m-2}^{m-1}\mu_{m0i}^{(f)}\bar{f}_{i,0}. \quad (17)$$

For $m = 1,\ldots,M$

$$g_{m,1}(\phi,\psi) = \sin(m\phi)\sin(\psi) - \sum_{i=0}^{M}\alpha_{m1i}^{(g)}\bar{f}_{i,0} - \sum_{i=1}^{m-1}\nu_{m1i}^{(g)}\bar{g}_{i,1}. \quad (18)$$

For $m = 0, \ldots, M$

$$f_{m,1}(\phi,\psi) = (\cos\psi)\bar{f}_{m,0}(\phi,\psi) - \sum_{i=0}^{M}\alpha_{m1i}^{(f)}\bar{f}_{i,0} - - \sum_{i=1}^{M}\gamma_{m1i}^{(f)}\bar{g}_{i,1} - \sum_{i=0}^{m-1}\mu_{m1i}^{(f)}\bar{f}_{i,1}. \quad (19)$$

For $m = 1, \ldots, M$

$$g_{m,2}(\phi,\psi) = (\cos\psi)\bar{g}_{m,1} - \sum_{i=0}^{M}\alpha_{m2i}^{(g)}\bar{f}_{i,0} - \sum_{i=0}^{M}\mu_{m2i}^{(g)}\bar{f}_{i,1} - \sum_{i=1}^{M}\gamma_{m2i}^{(g)}\bar{g}_{m,1} - \sum_{i=1}^{m-1}\nu_{m2i}^{(g)}\bar{g}_{i,2}. \quad (20)$$

For $n = 2, \ldots, N$,

For $m = 0, \ldots, M$,

$$f_{m,n}(\phi,\psi) = \cos(\psi)\bar{f}_{m,n-1} - \sum_{i=0}^{M}\alpha_{mni}^{(f)}\bar{f}_{i,n-1} - - \sum_{i=m}^{M}\beta_{mni}^{(f)}\bar{f}_{i,n-2} - \sum_{i=1}^{M}\gamma_{mni}^{(f)}\bar{g}_{i,n} - - \sum_{i=1}^{M}\delta_{mni}^{(f)}\bar{g}_{i,n-1} - \sum_{i=0}^{m-1}\mu_{mni}^{(f)}\bar{f}_{i,n} \quad (21)$$

end.

If $n < N$ *then*

For $m = 1, \ldots, M$,

$$g_{m,n+1}(\phi,\psi) = (\cos\psi)\bar{g}_{m,n} - \sum_{i=0}^{M}\alpha_{m,n+1,i}^{(g)}\bar{f}_{i,n-1} - - \sum_{i=1}^{M}\gamma_{m,n+1,i}^{(g)}\bar{g}_{i,n} - \sum_{i=m}^{M}\delta_{m,n+1,i}^{(g)}\bar{g}_{i,n-1} - - \sum_{i=0}^{M}\mu_{m,n+1,i}^{(g)}\bar{f}_{i,n} - \sum_{i=1}^{m-1}\nu_{m,n+1,i}^{(g)}\bar{g}_{i,n+1} \quad (22)$$

end.

end.

end.

Theorem 2 The set of functions

$$\{\{\{\bar{f}_{mn}(\phi,\psi)\}_{m=0}^{M}\}_{n=0}^{N}, \{\{\bar{g}_{mn}(\phi,\psi)\}_{m=1}^{M}\}_{n=1}^{N}\}$$

computed using Algorithm 2 is an orthonormal basis that spans the same space that the set of functions $\{\{\{\cos(m\phi)\cos(n\psi)\}_{m=0}^{M}\}_{n=0}^{N}$, $\{\{\sin(m\phi)\sin(n\psi)\}_{m=1}^{M}\}_{n=1}^{N}\}$ does.

Proof: The proof of this theorem can be found in [7].

Again, this theorem allows us to compute the orthonormal basis for the centro-symmetric case efficiently. Once the orthonormal basis has been determined, the actual frequency response can be obtained from

$$H(\phi,\psi) = \sum_{m=0}^{M}\sum_{n=0}^{N}\langle D(\phi,\psi), \bar{f}_{mn}(\phi,\psi)\rangle \bar{f}_{mn}(\phi,\psi) + \sum_{m=1}^{M}\sum_{n=1}^{N}\langle D(\phi,\psi), \bar{g}_{mn}(\phi,\psi)\rangle \bar{g}_{mn}(\phi,\psi). \quad (23)$$

The filter coefficients can be recovered from the frequency response using an inverse 2D FFT.

5. EXAMPLES

In order to demonstrate the applicability of the proposed algorithm, a number of filters have been designed for the quadrantally-symmetric case and for the centro-symmetric case as well.

Example 1: In the first example, we design a quadrantally-symmetric filter using Algorithm 1. The desired frequency response has rhombic shape. In this case, the weighting function is 5 over the passband and 1 over the stopband. The number of floating point operations required to compute the filter coefficients and the TSE for different filter sizes is depicted in table 1. In every case the number of frequency points was $L_\phi = L_\psi = 64$ from 0 to π. The actual frequency response for $M = N = 14$ is shown in figure 1.

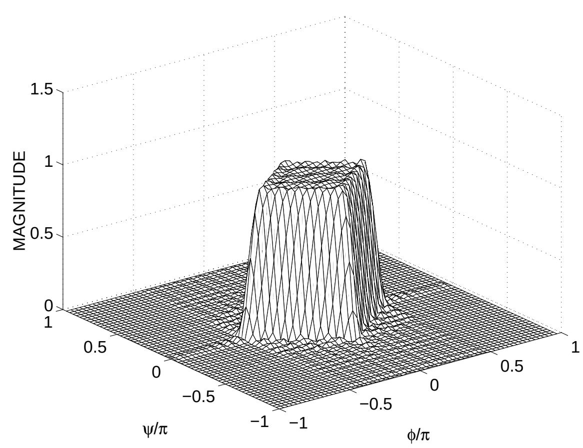

Figure 1: Frequency response for Example 1

Example 2: In the second example, we design a centro-symmetric filter using Algorithm 2. The passband has an elliptic shape rotated 30^o with respect to the horizontal axis. The major axis of the passband edge is 0.4π and the minor

Table 1: Number of floating point operations for Example 1

Filter size	ON basis method	General WLS method	TSE
$M = N = 13$	74770967	168705314	0.6095
$M = N = 14$	91552909	220894869	0.4291
$M = N = 15$	110673299	284598674	0.2875
$M = N = 16$	132284627	361464751	0.1946

Table 2: Number of floating point operations for Example 2

Filter size	ON basis method	General WLS method	TSE
$M = N = 12$	396318891	812961640	1.3078
$M = N = 13$	498552642	1.1008×10^9	0.8450
$M = N = 14$	617036783	1.4603×10^9	0.6016
$M = N = 15$	752972466	1.9029×10^9	0.4505

axis is 0.3π. The major axis of the stopband edge is 0.5π and the minor axis is 0.375π. In this case, the weighting function is 1 over the passband and 1 over the stopband. The number of floating point operations required to compute the filter coefficients and the TSE for different filter sizes is depicted in table 2. In every case the number of frequency points was $L_\phi = 64$ from 0 to π and $L_\psi = 128$ from $-\pi$ to π. The actual frequency response for $M = N = 14$ is shown in figure 2.

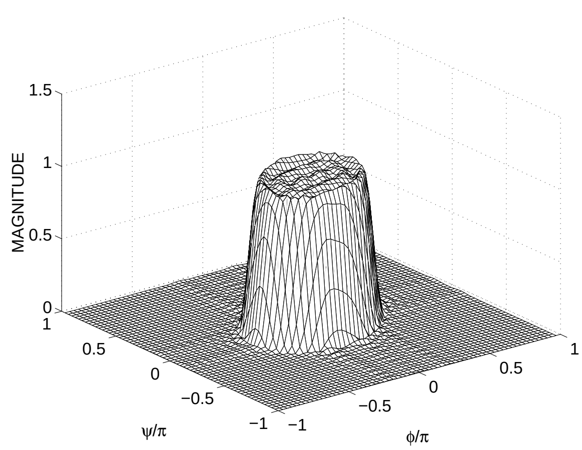

Figure 2: Frequency response for Example 2

6. CONCLUSIONS

In this paper, a fast algorithm for weighted least squares 2D linear-phase FIR filter design has been presented for both quadrantally-symmetric and centro-symmetric filters. It presents the problem of filter design as the problem of projecting a desired frequency response onto an appropriate subspace. An efficient way to compute an orthonormal basis that spans that subspace has been developed. By doing this, the usual matrix inversion involved in least squares filter design and the computational burden associated with it are avoided. Moreover, the special structure of the cosine basis allows for a substantial reduction of the amount of computation required to get the orthonormal basis. The applicability of the proposed algorithm has been demonstrated through examples.

7. REFERENCES

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