

# ITERATIVE NONLINEAR MMSE MULTIUSER DETECTION

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## ABSTRACT

This paper introduces the notion of nonlinear MMSE multiuser detection and shows that MMSE signal estimates followed by single user detectors yield MAP (or minimum probability of error) decisions for CDMA signals. Iterative solutions are proposed for nonlinear MMSE estimation. If interferers' codes are assumed to be known by the signal estimators for each user, the MMSE solution is shown to be a fixed point which is reached in one iteration, but is computationally intractable. If interferers' codes are not assumed to be known, the resulting iterative multiuser detector (called random-code NMIC) is shown to be of the same order of complexity as a conventional multistage interference canceler. Furthermore, the nonlinear MMSE solution is shown to be a fixed point of the random-code NMIC. Particular solutions are presented for the cases of BPSK and M-ary orthogonal spread spectrum systems. Audio demonstrations of random-code NMIC performance can be found at <http://www.nd.edu/~aspect/>.

## 1. INTRODUCTION

Optimal detection of signals in the presence of multiple access interference (MAI) has been a topic of intensive research activity due to its potentially vast benefits in CDMA systems. Maximum-likelihood and minimum probability of error detectors are known to be practically infeasible due to exponential computational complexity in the number of users. On the other hand, the efficacy of linear multiuser detectors is severely limited in the presence of coding, block modulation, multipath propagation, asynchronism and long spreading codes [1].

In this paper, we propose a novel multiuser detection strategy based on nonlinear minimum mean-squared error<sup>1</sup> (MMSE) estimation of the signals received from each of the users. Specifically, our objective is to compute MMSE (also maximum SNR) estimates of all the users' signals as functions of the composite received signal and the codes of all users. The MMSE signal estimates, when passed through minimum distance single user (i.e. matched filter) detectors, are shown to yield maximum *a posteriori* probability (MAP) decisions, i.e., minimum probability of error decisions, on the transmitted signals of all users.

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<sup>1</sup>The term nonlinear MMSE is used in this paper to distinguish it from the more widely used linear MMSE estimation. Nonlinear MMSE estimates can also be linear (e.g. Gaussian random variables).

Exponential computational complexity of MAP decisions also implies exponential complexity of exact computation of the nonlinear MMSE signal estimates. However, it is shown in this paper that the nonlinear MMSE signal estimates can be approximated using iterative algorithms. In this paper, we present a simple iterative algorithm derived by imposing an optimization procedure in a multistage interference cancellation receiver. An important result of this paper is to show that the nonlinear MMSE signal estimates are a fixed point of the proposed iterative algorithm. Furthermore, specific realizations of the iterative algorithm in the important cases of BPSK and M-ary orthogonally modulated CDMA systems are presented and are shown to be of the same complexity as conventional multistage interference cancelers (also called parallel interference cancelers (PIC)). M-ary orthogonal modulation is used on the uplink of the CDMA system specified by the IS-95 standard. The algorithms and propositions presented in this paper are valid for both synchronous and asynchronous multiple access systems.

The paper is organized as follows. Section 2 formulates the nonlinear MMSE multiuser detection problem and presents the general solution. It is shown in Section 2 that nonlinear MMSE estimates with single user detectors yield minimum probability of error decisions. Two iterative algorithms to approximate the nonlinear MMSE estimates are proposed in Section 3, namely, *deterministic-code NMIC* (nonlinear MMSE interference canceler) and *random-code NMIC*. Also, Section 3 proves that the nonlinear MMSE signal estimates are fixed points of both these algorithms. Practical realizations of the random-code NMIC are described in Section 4 for the cases of CDMA systems with BPSK or M-ary orthogonal modulation.

## 2. NONLINEAR MMSE MULTIUSER DETECTION

Consider a multiple access digital communication system in which  $K$  users are simultaneously transmitting signals to a central receiver. In a finite observation interval, all signals can be represented as vectors of their samples. These vectors can be obtained either as *chip-matched filter* outputs or simply by sampling the baseband received signal at a sufficiently large sampling rate. The received vector from the  $k$ th user can be written as  $\mathbf{r}_k = \mathbf{r}_k(\mathbf{m}_k, \mathbf{c}_k)$ , where  $\mathbf{m}_k$  is the information-bearing message,  $\mathbf{c}_k$  represents the channel and spreading codes, and  $\mathbf{r}_k(\cdot, \cdot)$  is a known function that depends on the coding (if any), modulation and spreading (if any) schemes, and channel characteristics. Message  $\mathbf{m}_k$  is

drawn from a finite message alphabet  $S_k$ . Both the message  $m_k$  and code  $\mathbf{c}_k$  are treated as random quantities. The composite received signal at the central receiver can be modeled as

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_K + \mathbf{v}, \quad (1)$$

where  $\mathbf{v}$  is zero-mean noise. We assume that  $\{\mathbf{r}_1, \dots, \mathbf{r}_K, \mathbf{v}\}$  are mutually independent.

The general problem of multiuser detection is to decide which of the possible signals was transmitted by each user, using the observation  $\mathbf{r}$  and knowledge of the codes  $C \triangleq [\mathbf{c}_1 \mathbf{c}_2 \cdots \mathbf{c}_K]$ . In lieu of directly computing the MAP or ML decisions for the transmitted signals [2, 3], we sought to compute *estimates* of the signals received from each user,  $X = [\hat{\mathbf{r}}_1 \hat{\mathbf{r}}_2 \cdots \hat{\mathbf{r}}_K]$  as a function of  $(\mathbf{r}, C)$ , in a way that minimizes the MAI component in each user's signal estimate. Final decisions on the transmitted signals are made by using a bank of single user detectors that operate on these estimates with reduced MAI. If square of the distance between two random vectors is measured in terms of the mean-squared Euclidean 2-norm of their difference, an optimal estimation strategy is to compute  $X$  such that for each  $k$ ,  $\hat{\mathbf{r}}_k$  is closest to  $\mathbf{r}_k$  (or, equivalently, by maximizing the SNR in  $\hat{\mathbf{r}}_k$ ) among all functions of  $(\mathbf{r}, C)$ . In other words, the objective is to obtain the nonlinear MMSE estimate of the  $k$ th user's signal  $\mathbf{r}_k$ ,  $k = 1, \dots, K$ , given the observation  $(\mathbf{r}, C)$ . We denote the nonlinear MMSE estimate by  $X_* = [\hat{\mathbf{r}}_{*1} \hat{\mathbf{r}}_{*2} \cdots \hat{\mathbf{r}}_{*K}]$  where for  $k = 1, \dots, K$ ,

$$\hat{\mathbf{r}}_{*k} \triangleq \arg \min_{\hat{\mathbf{r}}_k = \hat{\mathbf{r}}_k(\mathbf{r}, C)} E\{\|\mathbf{r}_k - \hat{\mathbf{r}}_k\|^2\}. \quad (2)$$

The solution to the nonlinear MMSE estimation problem is the well-known conditional mean estimate (see, e.g., [4]), given by

$$\hat{\mathbf{r}}_{*k} = E\{\mathbf{r}_k | \mathbf{r}, C\}, \quad k = 1, \dots, K. \quad (3)$$

Besides being intuitively appealing, the nonlinear MMSE signal estimates facilitate MAP decisions on the transmitted signals using simple single user detectors, as described in the following observation. Note that this result is true in both synchronous and asynchronous scenarios.

**Proposition 1** *Assume that the received signal from each user is BPSK modulated or M-ary orthogonal modulated with direct sequence spreading. Then the nonlinear MMSE estimate  $\hat{\mathbf{r}}_{*k}$  followed by a minimum distance single user detector yields the MAP decision for the  $k$ th user's signal, that is,*

$$\arg \min_{m \in S_k} \|\mathbf{r}_k(m, \mathbf{c}_k) - \hat{\mathbf{r}}_{*k}\|^2 = \arg \max_{m \in S_k} P[m_k = m | \mathbf{r}, C]. \quad (4)$$

*Sketch of Proof:* Let  $\mathbf{r}_{k,i}$  be a sub-vector of  $\mathbf{r}_k$  corresponding to the  $i$ th symbol of user  $k$ , and let  $\mathbf{c}_{k,i}$  be the corresponding sub-vector of  $\mathbf{c}_k$ . We assume that different symbols of the same user are statistically independent. It can be shown that among all permissible user  $k$ 's received vectors, the one closest to  $\hat{\mathbf{r}}_{*k}$  is the one whose sub-vectors  $\mathbf{r}_{k,i}$  are closest to the corresponding sub-vectors in  $\hat{\mathbf{r}}_{*k}$  for all  $i$ . Also, the MAP decision for  $\mathbf{r}_k$  is composed of MAP decisions for individual symbols in  $\mathbf{r}_k$ . Therefore, it is sufficient to prove the equivalence of minimum distance from the MMSE estimate and MAP decision for a single arbitrary symbol in  $\mathbf{r}_k$ . If the signals are BPSK modulated, then

$$\begin{aligned} \hat{\mathbf{r}}_{*k,i} &= E\{\mathbf{r}_{k,i} | \mathbf{r}, C\} = A_k \mathbf{c}_{k,i} E\{b_{k,i} | \mathbf{r}, C\} \\ &= A_k \mathbf{c}_{k,i} \{P[b_{k,i} = +1 | \mathbf{r}, C] \\ &\quad - P[b_{k,i} = -1 | \mathbf{r}, C]\}, \end{aligned} \quad (5)$$

where  $A_k$  is the  $k$ th user's received amplitude and  $b_{k,i} = \pm 1$  is the  $i$ th bit. It is clear from the above that  $\hat{\mathbf{r}}_{*k,i}$  is closer to  $A_k \mathbf{c}_{k,i}$  than to  $-A_k \mathbf{c}_{k,i}$  if, and only if,  $P[b_{k,i} = +1 | \mathbf{r}, C] > P[b_{k,i} = -1 | \mathbf{r}, C]$ . This concludes the proof for the BPSK case. If the signal  $\mathbf{r}_{k,i}$  is drawn (after spreading) from an M-ary orthogonal signal set  $S = \{\mathbf{s}_1, \dots, \mathbf{s}_M\}$  with equal energies, then  $\hat{\mathbf{r}}_{*k,i}$  is closest to  $\mathbf{s}_j$  among all signals in  $S$  if, and only if, its correlation with  $\mathbf{s}_j$  is maximum. The MMSE estimate of  $\mathbf{r}_{k,i}$  can be written as

$$\hat{\mathbf{r}}_{*k,i} = \sum_{j=1}^M \mathbf{s}_j P[\mathbf{r}_{k,i} = \mathbf{s}_j | \mathbf{r}, C] \quad (6)$$

Therefore, the correlation of  $\hat{\mathbf{r}}_{*k,i}$  with  $\mathbf{s}_j$  is

$$\langle \hat{\mathbf{r}}_{*k,i}, \mathbf{s}_j \rangle \propto P[\mathbf{r}_{k,i} = \mathbf{s}_j | \mathbf{r}, C] \quad (7)$$

Equation (7) implies that  $\hat{\mathbf{r}}_{*k,i}$  is closest to  $\mathbf{s}_j$  if, and only if,  $P[\mathbf{r}_{k,i} = \mathbf{s}_j | \mathbf{r}, C]$  is the largest. This concludes the proof.

Proposition 1 implies that the MAP decision for the  $k$ th user (that minimizes the  $k$ th user's probability of error) can be obtained by computing the MMSE signal estimate  $\hat{\mathbf{r}}_{*k}$  followed by a conventional minimum distance single user detector for the  $k$ th user. If we can compute (or approximate)  $X_*$  with reasonable complexity, this is practically attractive since minimum distance single user detectors are extremely simple to implement. In the case of BPSK spread spectrum signals, for instance, the minimum distance detector is a matched filter (matched to the desired user's spreading code) followed by a hardlimiter.

We know from Verdú's work on optimal multiuser detection (see, e.g., [3] and references therein) that minimum probability of error detection is exponentially complex in  $K$ , the number of users. Since the complexity of single user detection is linear in  $K$ , this along with Proposition 1 implies that exact computation of the nonlinear signal estimate  $X_*$  is also exponentially complex in  $K$ . Therefore, instead of computing the exact MMSE estimate  $X_*$ , we seek iterative solutions to approximate it.

### 3. ITERATIVE SOLUTIONS FOR NONLINEAR MMSE MULTIUSER DETECTION

An extension to the idea of multistage interference cancellation (see, e.g., [5]) to include an optimization procedure at each stage is used here to approximate the nonlinear MMSE signal estimate  $X_*$ . In each iteration (stage) of a general multistage interference canceler, past estimates of all the signals are used to subtract estimated interference from each user's signal estimate. Let  $X_n = [\hat{\mathbf{r}}_1^n \hat{\mathbf{r}}_2^n \cdots \hat{\mathbf{r}}_K^n]$  denote the signal estimate after the  $n$ th iteration. The iterations are started with  $\hat{\mathbf{r}}_k^0 = \mathbf{r}$ ,  $k = 1, \dots, K$ . If  $\mathcal{X}(\mathbf{r}, C)$  is the space of all random matrices of the same size as  $X_*$  that are functions of  $(\mathbf{r}, C)$ , then the canceler defines a mapping  $\mathcal{M} : \mathcal{X}(\mathbf{r}, C) \rightarrow \mathcal{X}(\mathbf{r}, C)$  such that  $X_n = \mathcal{M}(X_{n-1})$ . A canonical mapping  $\mathcal{M}$  for any multistage interference canceler is defined by two operations, namely, *estimation* and *cancellation*. The estimation operation involves computing *a priori* signal estimates  $\hat{\mathbf{r}}_k^{n|n-1}$  from the *a posteriori* estimates  $\hat{\mathbf{r}}_k^{n-1}$  of the previous iteration. The cancellation operation cancels the *a priori* interference estimates as follows. For all  $k = 1, \dots, K$ ,

$$\hat{\mathbf{r}}_k^n = \mathbf{r} - \sum_{i \neq k} \hat{\mathbf{r}}_i^{n|n-1}, \quad (8)$$

In a conventional multistage interference canceler, the estimation operation is described by  $\hat{\mathbf{r}}_k^{n|n-1} = \mathbf{r}_k(\hat{m}_k^{n-1}, \mathbf{c}_k)$ , where  $\hat{m}_k^{n-1}$

is a hard decision on the transmitted symbol using a single user detector (such as a matched filter detector) on  $\hat{\mathbf{r}}_k^{n-1}$ . This choice of estimation procedure for each stage, however, leads to incorrect cancellation of interference whenever a single user detector makes an incorrect decision. Such cancellation errors are propagated to the following stages of the canceler.

Since approximations to the conditional mean signal estimates are required, we propose the following optimal estimation operation:

$$\hat{\mathbf{r}}_k^{n|n-1} = E\{\mathbf{r}_k | \hat{\mathbf{r}}_k^{n-1}, C\}. \quad (9)$$

The codes of all users are assumed known here for the estimation operation. Therefore, we call the resulting multistage interference canceler a deterministic-code NMIC. The following property of the deterministic-code NMIC follows immediately.

**Proposition 2** *The nonlinear MMSE signal estimate  $X_*$  is a fixed point of the deterministic-code NMIC, and the fixed point is reached in one iteration.*

We use the following lemma to prove Proposition 2. Proof of the lemma is omitted due to lack of space.

**Lemma 1** *If  $X, Y$  and  $Z$  are random variables, then*

$$E\{X|E\{X|Y, Z\}, Z\} = E\{X|Y, Z\}. \quad (10)$$

To see that  $X_*$  is a fixed point of  $\mathcal{M}$  defined by (8) and (9), let  $X_{n-1} = X_*$ . From (9) and (10) it follows that

$$\begin{aligned} \hat{\mathbf{r}}_k^{n|n-1} &= E\{\mathbf{r}_k | \hat{\mathbf{r}}_{*k}, C\} = E\{\mathbf{r}_k | E\{\mathbf{r}_k | \mathbf{r}, C\}, C\} \\ &= E\{\mathbf{r}_k | \mathbf{r}, C\} = \hat{\mathbf{r}}_{*k}. \end{aligned} \quad (11)$$

Further noting that  $\sum_{k=1}^K \hat{\mathbf{r}}_{*k} = \mathbf{r}$ , we can conclude from (8) that  $X_n = X_* = X_{n-1}$ . That convergence is attained in one iteration can be inferred by using the fact that  $\hat{\mathbf{r}}_k^0 = \mathbf{r}$ ,  $k = 1, \dots, K$ , in (8) and (9), with  $n = 1$ .

Note that computation of  $X_*$  is involved in the first iteration of the deterministic-code NMIC. Therefore, the above description of the deterministic-code NMIC only illustrates the convergence property of the multistage cancellation structure, rather than providing a way to compute  $X_*$ . More importantly, it leads us to a practically feasible NMIC by relaxing the requirement to use knowledge of all users' codes in the estimation step. In particular, we assume for the purpose of computing  $\hat{\mathbf{r}}_k^{n|n-1}$  that all interfering users' codes are unknown and random, that is,

$$\hat{\mathbf{r}}_k^{n|n-1} = E\{\mathbf{r}_k | \hat{\mathbf{r}}_k^{n-1}, \mathbf{c}_k\}. \quad (12)$$

We call this the random-code NMIC. (Note that the conventional multistage interference canceler also does not use knowledge of interferers' codes to compute  $\hat{\mathbf{r}}_k^{n|n-1}$ .) The following remarkable property of the random-code NMIC, combined with its ease of implementation, makes it an attractive iterative scheme to approximate the nonlinear MMSE signal estimate  $X_*$ .

**Proposition 3** *The nonlinear MMSE signal estimate  $X_*$  is a fixed point of the random-code NMIC.*

The proof of this result also uses Lemma 1, and is very similar to that of Proposition 2.

The above proposition indicates that the MMSE signal estimate  $X_*$  that is a function of all the users' codes is a fixed point to

the NMIC that does not require explicit knowledge of signal cross-correlations in each iteration. Since signal cross-correlations need not be computed, the following section shows that the random-code NMIC is particularly simple to implement when the number of users and spreading gain are large. The realizations of random-code NMIC in two important cases—namely, direct sequence CDMA systems with BPSK modulation and M-ary orthogonal modulation—are considered next.

#### 4. REALIZATIONS OF RANDOM-CODE NMIC

The only operation that needs to be specified to define a random-code NMIC is the estimation operation (12), since the cancellation operation (8) is identical in all multistage interference cancelers. As noted in the proof of Proposition 1,  $\mathbf{r}_{k,i}$  is the sub-vector of  $\mathbf{r}_k$  that corresponds to the  $i$ th symbol. Since the conditional mean estimate of  $\mathbf{r}_k$  is comprised of conditional mean estimates of  $\mathbf{r}_{k,i}$  for all  $i$ , the estimation step can be broken down into symbol-by-symbol estimation. We can therefore restrict our attention to computing  $\hat{\mathbf{r}}_{k,i}^{n|n-1}$  for an arbitrary symbol  $i$ . For convenience, we omit the subscript  $i$  in the sequel.

In a direct sequence CDMA system that employs BPSK modulation,  $\mathbf{r}_k = A_k b_k \mathbf{c}_k$ , where  $b_k \in \{-1, +1\}$ . It is shown in [6] that

$$\hat{\mathbf{r}}_k^{n|n-1} = A_k \mathbf{c}_k E\{y_k^n\}, \quad (13)$$

where  $y_k^n$  is the output of the matched filter for user  $k$  with  $\hat{\mathbf{r}}_k^{n-1}$  as input. If the number of users and the spreading gain are large, then  $y_k^n$  conditioned on  $b_k$  can be modeled as a Gaussian random variable [6] with mean  $\tilde{A}_k^n b_k$  and variance  $\sigma_{k,n}^2$ . The effective received amplitude  $\tilde{A}_k^n$  is such that  $\tilde{A}_k^1 = A_k$ , but  $\tilde{A}_k^n < A_k$  for  $n > 1$ . The effective amplitude is determined by the correlation that exists between the desired signal and interference at the output of the matched filter in the  $n$ th iteration. The estimation operation can then be computed as

$$\hat{\mathbf{r}}_k^{n|n-1} = A_k \mathbf{c}_k \tanh(\tilde{A}_k^n y_k^n / \sigma_{k,n}^2), \quad (14)$$

where  $\sigma_{k,n}^2$  is the variance of interference plus noise at the output of the  $k$ th user's matched filter in the  $n$ th iteration. The use of sigmoidal nonlinearity of the form of (14) for interference cancellation has been proposed in [7] in a different context. A schematic of the estimation operation is shown in Figure 1. The parameters of the sigmoidal nonlinearity in (14), namely,  $\tilde{A}_k^n$  and  $\sigma_{k,n}^2$ , can be estimated adaptively [6].

In a direct sequence CDMA systems with M-ary orthogonal modulation, let the transmitted signal constellation (after spreading) of the  $k$ th user be  $\mathcal{Q}_k = \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_M\}$ , where  $\|\mathbf{s}_i\|^2 = 1$  for all  $i$  and  $\mathbf{s}_i \perp \mathbf{s}_j$  for all  $i \neq j$ . The  $k$ th user's received signal is given by  $\mathbf{r}_k = a_k \mathbf{x}_k$ , where the transmitted symbol  $\mathbf{x}_k \in \mathcal{Q}_k$ , and the complex received amplitude of user  $k$  is defined as  $a_k = A_k e^{j\phi_k}$  where  $A_k$  is the received amplitude and  $\phi_k$  is the received carrier phase of user  $k$ . As in the case of BPSK signals, it can be shown that

$$\hat{\mathbf{r}}_k^{n|n-1} = a_k E\{\mathbf{x}_k | \mathbf{y}_k^n\}, \quad (15)$$

where  $\mathbf{y}_k^n$  is the vector matched filter output in the  $n$ th iteration. It is comprised of correlations of  $\hat{\mathbf{r}}_k^{n-1}$  with all  $M$  orthogonal signals in the constellation. Again, for large number of users and spreading gain, the matched filter output conditioned on the transmitted symbol  $\mathbf{x}_k$  can be modeled as a Gaussian vector with mean  $\tilde{a}_k^n \mathbf{x}_k$ ,

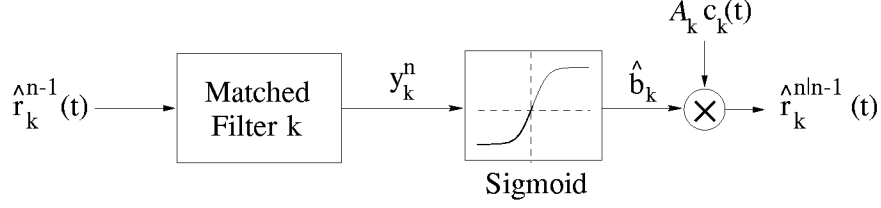


Figure 1: Estimation operation in a random-code NMIC for BPSK modulation

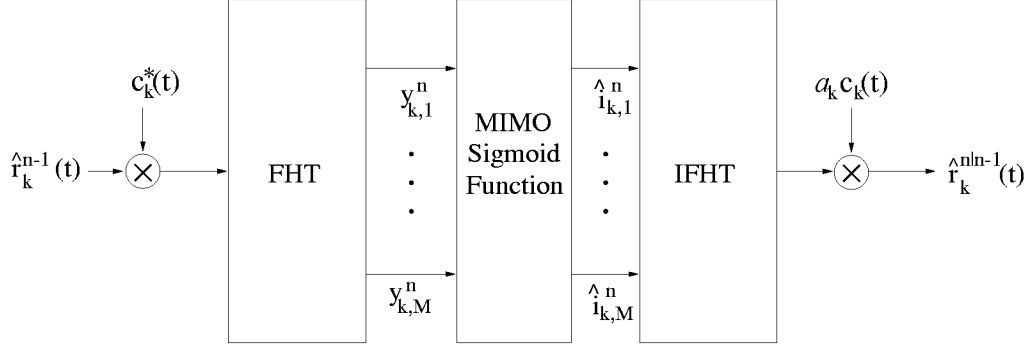


Figure 2: Estimation operation in a random-code NMIC for M-ary orthogonal modulation

where  $\tilde{a}_k^n$  is the *effective* complex-valued received amplitude. The estimation step can be computed as [6],

$$\hat{r}_k^{n|n-1} = a_k \frac{\sum_{j=1}^M z_j s_j}{\sum_{j=1}^M z_j}, \quad (16)$$

where

$$z_j \doteq \exp[\text{Re}\{(\tilde{a}_k^n)^* \mathbf{y}_{k,j}^n\} / \sigma_{k,n}^2], \quad (17)$$

$\mathbf{y}_{k,j}^n$  being the  $j$ th element of  $\mathbf{y}_k^n$ . If Walsh-Hadamard orthogonal sequences are used, the estimation operation can be performed efficiently using the Fast Hadamard Transform (FHT) and the Inverse Fast Hadamard Transform (IFHT) as shown in Figure 2. In the figure, multiplication by the complex conjugate of the spreading code and computation of signal correlations via FHT constitute matched filtering. This is followed by a multi-input-multi-output (MIMO) sigmoidal nonlinearity to compute  $\hat{b}_{k,j}^n = z_j / \sum_{i=1}^M z_i$ . These are used by the IFHT and respreaders to compute  $\hat{r}_k^{n|n-1}$ . As before, the sigmoidal parameters can be estimated adaptively.

The computational complexity of random-code NMIC can be seen from above to be of the same order as that of conventional PICs, in both BPSK and M-ary orthogonal modulation cases. Audio demonstrations from a simulated CDMA system using random-code NMIC and conventional interference cancellation schemes can be found at the web site [8].

## 5. CONCLUSIONS

A practically feasible iterative algorithm to approximate nonlinear MMSE signal estimates in the presence of MAI was presented in this paper. This algorithm can be used to compute approximate minimum probability of error decisions on the transmitted signals. Future work in this area includes convergence analysis of random-code NMIC, realizations of random-code NMIC for coded data

and in the presence of multiple antennas, and performance evaluation of random-code NMIC.

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