FRAMES IN ROTATED TIME-FREQUENCY PLANES

Aykut Bultan and Ali N. Akansu

Electrical and Computer Eng. Dept.,
New Jersey Center for Multimedia Research, NJIT, Newark, NJ 07102
e-mail: bultan@oak.njit.edu

ABSTRACT

Weyl-Heisenberg frames are complete signal representations corresponding to rectangular tiling of the time-frequency plane. Extensions of these frames are obtained in the rotated time-frequency planes by using the fractional Fourier transformation. It is shown that rotation does not affect the frame bounds. For some specific angles, lattices in rotated coordinates will map to the lattices in the Cartesian coordinates. The rotated Weyl-Heisenberg frames obtained are more suitable for chirp-like signal analysis and synthesis.

1. INTRODUCTION

Weyl-Heisenberg frames (WHFs) constitute the basic theory behind the discrete short-time Fourier transform (STFT), or in particular Gabor expansion [1]. Decomposition of signals into these frames provide valuable information about the local structures of the signal in the time-frequency plane. However, for signals that have chirp-like components, WHFs can not provide compact representations. In order to represent such signals effectively, frames in rotated coordinates are developed. Indeed, the most compact and efficient way to represent these signals is to use the chirplet decomposition [2], [3], [4]. However, in cases where computation time is critical, one may rather prefer to use the rotated frames. Different kinds of angular tilings of the time-frequency plane has been shown in [5].

The rotated Weyl-Heisenberg frames (RWHFs) are obtained by applying the fractional Fourier transform (FRFT) operator to the WHFs. Using the unitary property of the FRFT, it is shown that rotation does not change the frame bounds. At some discrete angles, a lattice mapping from rotated to Cartesian coordinates is obtained. Whenever these angles are used, computationally more efficient form of the rotated frames are obtained.

2. WEYL-HEISENBERG FRAMES AND THE FRACTIONAL FOURIER TRANSFORM

The set of the time-frequency shifted window functions $\{g_{q,p}(t)\}_{(q,p)\in\mathbf{Z^2}}$ constitutes a WHF if and only if for any $f(t)\in\mathbf{L^2(R)}$ [1],

$$A\|f\|^2 \leq \sum_{q=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} |< f, g_{q,p} >|^2 \leq B\|f\|^2 \qquad (1)$$

is satisfied where $g_{q,p}(t) \triangleq g(t-q\Delta c)e^{jp\Delta dt}$ and $A>0, B<\infty, \Delta c, \Delta d > 0$. If $\{g_{q,p}(t)\}$ constitutes a frame, then there exists a dual frame denoted as $\{\tilde{g}_{q,p}(t)\}$ such that any $f(t)\in\mathbf{L^2(R)}$ can be written as

$$f(t) = \sum_{q=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} < f, g_{q,p} > \tilde{g}_{q,p}(t). \qquad (2)$$

In [1], series computation of $\tilde{g}(t)$ from $g(t)$ is described. The oversampling ratio is given by $R = 2\pi/(\Delta c\Delta d)$, and it should be greater than or equal to 1 to obtain a complete representation. $R=1$ corresponds to critical sampling or Nyquist rate. To obtain the RWHFs from this set, we need the FRFT operator. The FRFT of a function $f(t)$ is defined in [6] as,

$$\begin{aligned} f_{-\alpha}(x) &\triangleq (\Gamma_{-\alpha}f)(x) \\ &\triangleq \sqrt{\frac{1-j\cot\alpha}{2\pi}}e^{j\frac{\cot\alpha}{2}x^2}\int_{-\infty}^{\infty} f(\tau)e^{j\frac{\cot\alpha}{2}\tau^2} \\ &\quad \times e^{-j\csc\alpha\tau x}d\tau \end{aligned} \qquad (3)$$

where Γ_α is the rotation operator. The minus sign is put in this equation to define the counter clockwise direction in the time-frequency plane as the positive direction of rotation. Using the rotation operator in $t-\omega$ plane results in a confusion of manipulating time and frequency variables of different units as having the same unit. As an example, in Eqn. (8), time and frequency variables are added as if they have the same unit. In order to solve this problem, a method of normalization has been proposed in [7]. Defining the normalization scale

$$\mathcal{S} \triangleq \sqrt{\frac{\text{Duration of the signal}}{\text{Bandwidth of the signal}}}, \quad \frac{\text{sec}}{\sqrt{\text{rad}}} \qquad (4)$$

then the normalized time-frequency variables are given as

$$t_n \triangleq \frac{t}{\mathcal{S}}, \ \sqrt{\text{rad}} \tag{5}$$

$$\omega_n \triangleq \mathcal{S}\omega, \ \sqrt{\text{rad}} \tag{6}$$

After this normalization the duration and bandwidth of the signal have the same length and the same unit. In this work, it is assumed that this dimensional normalization is performed beforehand.

3. FRAMES IN ROTATED COORDINATES

We define $s \triangleq \sqrt{\frac{\Delta c}{\Delta d}}$, and restrict s to be $s \in \mathbf{Z}^+$. Then the following lemma defines a lattice mapping from rotated to Cartesian coordinates, as illustrated in Fig. 1. A specific case of this lattice for $\alpha_m = \pi/4$ has been recently used for Gabor expansion [8], [9].

Lemma 1 *Let the discrete angles α_m be defined as*

$$\alpha_m \triangleq \begin{cases} \arctan(\frac{m}{s^2}), & m \in I_1 \\ \frac{\pi}{2} - \arctan(\frac{m_1}{s^2}), & m \in I_2 \\ -\frac{\pi}{2} + \arctan(\frac{m_2}{s^2}), & m \in I_3 \end{cases} \tag{7}$$

where $m_1 = 2s^2 - m$, $m_2 = 2s^2 + m$, and $m \in \mathbf{Z}$. The intervals are given by $I_1 = [-s^2, s^2]$, $I_2 = (s^2, 2s^2)$, and $I_3 = (-2s^2, -s^2)$. Then the projections of rotated lattice points $(q\Delta c, p\Delta d)$ in $t_{\alpha_m} - \omega_{\alpha_m}$ plane constitute a rectangular lattice in $t - \omega$ plane whose indexes are given by $(l(q,p)\Delta u, k(q,p)\Delta\nu)$ where

$$\Delta u = \begin{cases} \Delta d/\sqrt{m^2+s^4}, & m \in I_1 \\ \Delta c/\sqrt{{m_1}^2+s^4}, & m \in I_2 \\ \Delta c/\sqrt{{m_2}^2+s^4}, & m \in I_3 \end{cases}$$

$$\Delta \nu = \begin{cases} \Delta c/\sqrt{m^2+s^4}, & m \in I_1 \\ \Delta d/\sqrt{{m_1}^2+s^4}, & m \in I_2 \\ \Delta d/\sqrt{{m_2}^2+s^4}, & m \in I_3 \end{cases}$$

$$l(q,p) = \begin{cases} s^4 q - mp, & m \in I_1 \\ m_1 q - p, & m \in I_2 \\ m_2 q + p, & m \in I_3 \end{cases}$$

$$k(q,p) = \begin{cases} mq - p, & m \in I_1 \\ qs^4 + m_1 p, & m \in I_2 \\ -qs^4 + m_2 p, & m \in I_3 \end{cases}$$

and $\Delta u, \Delta\nu \in \mathbf{R}^+$, $k(q,p), l(q,p) \in \mathbf{Z}$.

Proof: A point $(t_\alpha, \omega_\alpha)$ in the rotated coordinates is mapped to a point (t, ω) in the Cartesian coordinates by a linear transformation

$$\begin{aligned} t &= t_\alpha \cos\alpha - \omega_\alpha \sin\alpha \\ \omega &= t_\alpha \sin\alpha + \omega_\alpha \cos\alpha. \end{aligned} \tag{8}$$

If the time-frequency centers of the atoms in the rotated coordinates are constructed like the WHFs i.e., $(t_\alpha, \omega_\alpha) = (q\Delta c, p\Delta d)$, then the projections of these points on the Cartesian coordinates are given from Eqn. (8) as

$$\begin{aligned} t &= q\Delta c \cos\alpha - p\Delta d \sin\alpha \\ \omega &= q\Delta c \sin\alpha + p\Delta d \cos\alpha. \end{aligned} \tag{9}$$

i) $\alpha_m = \arctan(\frac{m}{s^2})$ for $-s^2 \leq m \leq s^2$, then one can write

$$\cos\alpha_m = \frac{s^2}{\sqrt{m^2+s^4}}, \quad \sin\alpha_m = \frac{m}{\sqrt{m^2+s^4}}. \tag{10}$$

Putting these into Eqn. (9), and reordering of the terms yield

$$t = q\Delta c \frac{s^2}{\sqrt{m^2+s^4}} - p\Delta d \frac{m}{\sqrt{m^2+s^4}} \tag{11}$$

$$\omega = q\Delta c \frac{m}{\sqrt{m^2+s^4}} + p\Delta d \frac{m}{\sqrt{m^2+s^4}}. \tag{12}$$

Parts *ii)* and *iii)* can be proved easily with similar manipulations.

It can be easily shown that this lemma can be generalized to include the case $1/s \in \mathbf{Z}^+$. After finding the pairs $(l(q,p)\Delta u, k(q,p)\Delta\nu)$, they can be reordered such that $l \in \mathbf{L} = \{l(q,p)\}_{q,p}$ and $\mathbf{K}_l = \{k(q',p') : l = l(q',p')\}$. Then the reordered points can be written as (l, k), where $l \in \mathbf{L}$ and $k \in \mathbf{K}_l$. We define

$$g_m(t) \triangleq (\Gamma_{\alpha_m} g)(t) \tag{13}$$

$$g_{m,l,k}(t) \triangleq g_m(t - l\Delta u)e^{jk\Delta\nu t} \tag{14}$$

similar definitions are valid for the dual frame $\tilde{g}(t)$ also.

Theorem 1 *Let $g_{q,p}(t)$ constitute a Weyl-Heisenberg frame with frame bounds A and B and with the dual frame $\tilde{g}_{q,p}(t)$, then*

i) For each α_m fixed, $\{g_{m,l,k}(t)\}_{l,k}$ constitutes a frame with frame bounds A and B.

ii) Dual frame of $g_{m,l,k}(t)$ is given by $\tilde{g}_{m,l,k}(t)$.

Proof: i) Since $g_{q,p}(t)$ is a frame with bounds A and B, for any $f(t) \in \mathbf{L^2(R)}$ the following can be written

$$A||f||^2 \leq \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} |< f, g_{q,p} >|^2 \leq B||f||^2, \tag{15}$$

where

$$< f, g_{q,p} >= \int_{-\infty}^{\infty} f(t) g^*(t - q\Delta c)e^{-jp\Delta dt} dt. \tag{16}$$

Since the rotation operator is inner product preserving, we can write $< f, g_{q,p} >=< \Gamma_{\alpha_m} f, \Gamma_{\alpha_m} g_{q,p} >$. Defining $f_m(t) \triangleq (\Gamma_{\alpha_m} f)(t)$ this becomes

$$< f, g_{q,p} >=< f_m, e^{j\theta} g_{m,l,k} >, \tag{17}$$

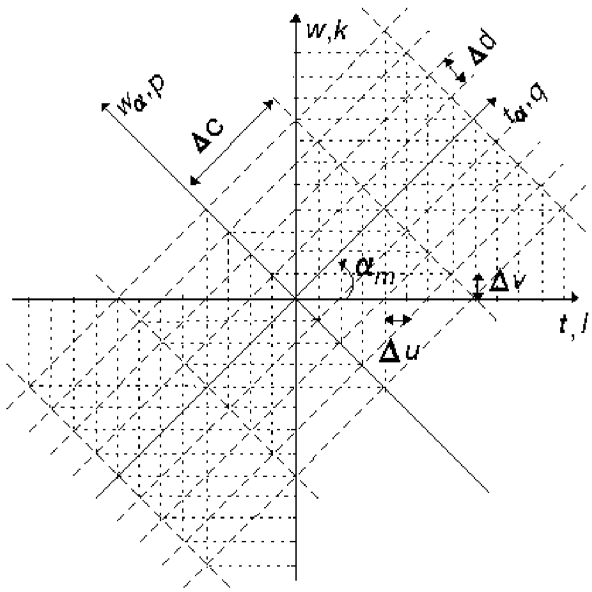

Figure 1: Lattice mapping for $\alpha_m = 45°$.

where $g_{m,l,k}(t)$ is defined in Eqn. (14), and θ is found as $\theta = \frac{\sin\alpha_m \cos\alpha_m}{2}(p^2\Delta d - q^2\Delta c) + qp\Delta c\Delta d \sin^2\alpha_m$.

Since rotation operation is inner product preserving, then $||f_m|| = ||f||$. Putting Eqn. (17) into Eqn. (16), and reordering of the summation indexes yield

$$A||f||^2 \leq \sum_{k\in\mathbf{K}_l}\sum_{l\in\mathbf{L}} |< f, g_{m,l,k} >|^2 \leq B||f||^2, \qquad (18)$$

which means $g_{m,l,k}(t)$ constitutes a frame for each α_m with frame bounds A and B.

ii) Since $\tilde{g}_{q,p}(t)$ is the dual frame of $g_{q,p}(t)$, then any $f(t) \in \mathbf{L^2}(\mathbf{R})$ can be written as

$$f(t) = \sum_{q=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} < f, g_{q,p} > \tilde{g}_{q,p}(t). \qquad (19)$$

Transforming both sides by Γ_{α_m} and using Eqn. (17) yields

$$f_m(t) = \sum_{k\in\mathbf{K}_l}\sum_{l\in\mathbf{L}} < f_m, g_{m,l,k} > \tilde{g}_m(t - l\Delta u)e^{jk\Delta\nu t}. \qquad (20)$$

Therefore, one can write for any $f(t) \in \mathbf{L^2}(\mathbf{R})$

$$f(t) = \sum_{k\in\mathbf{K}_l}\sum_{l\in\mathbf{L}} < f, g_{m,l,k} > \tilde{g}_{m,l,k}(t). \qquad (21)$$

This means, $\tilde{g}_{m,l,k}$ is the dual frame of $g_{m,l,k}$ for each α_m. The rotated frame of Gaussians are shown in Fig. 2.

Example: As an example, window function is chosen as Gaussian $g(t) = \frac{1}{2\pi^{1/4}}e^{-\frac{t^2}{32}}$. The redundancy ratio is set to $R = 4$. The time and frequency discrete step lengths are

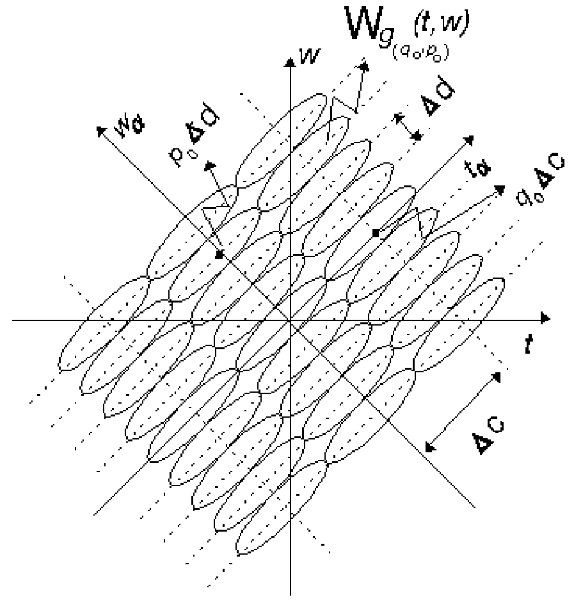

Figure 2: Rotated frame of Gaussians. Contour plots of the Wigner distributions of the time-frequency shifted windows are shown.

chosen as $\Delta c = 4\sqrt{\frac{\pi}{2}}$ and $\Delta d = \frac{1}{4}\sqrt{\frac{\pi}{2}}$, respectively. The rotated window $g_m(t)$ for $\alpha_m = \pi/4$ is found by applying the FRFT to the Gaussian which yields,

$$g_m(t) = \frac{e^{-j0.362}}{64.25\pi^{1/4}}e^{-(0.062-j0.029)t^2}. \qquad (22)$$

The sampling rate is chosen as $\Delta t = \sqrt{\frac{2\pi}{N}}$, where $N = 256$ is the signal length. In Fig. 3, the Gaussian, its rotated and time frequency shifted versions are shown. The Wigner distribution of $g_{m,l,k}(t)$ is given in Fig. 4.

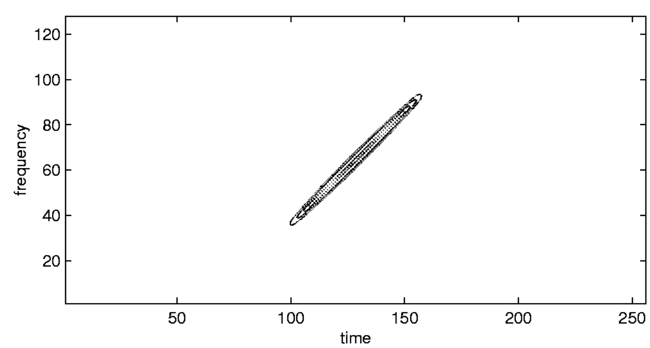

Figure 4: The Wigner distribution of $g_{m,l,k}(t)$

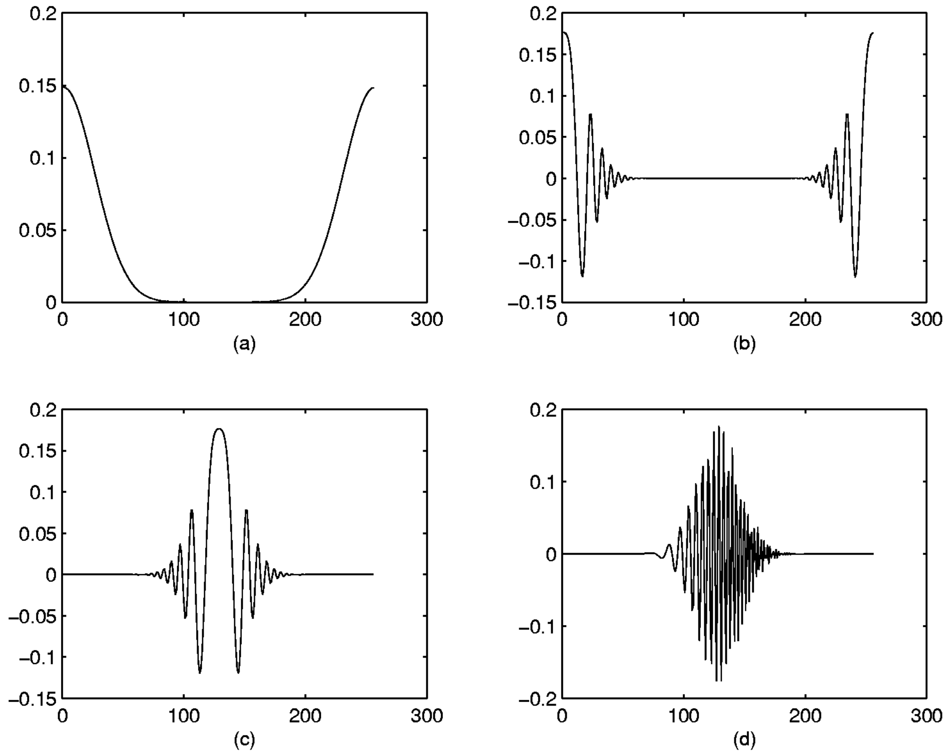

Figure 3: (a)The Gaussian function $g(t)$, (b) $g_m(t)$, (c) $g_m(t - l\Delta u)$, (d) $g_{m,l,k}(t) = g_m(t - l\Delta u)e^{jk\Delta\nu t}$.

4. CONCLUSIONS

In this paper, rotated frames are developed from the WHFs by using the FRFT. First a lattice mapping from rotated to Cartesian coordinates is given, then the rotated frames are introduced. It is shown that, the rotation does not change the frame bounds. These frames are important for the analysis of chirp-like energy distributions in the time-frequency plane. They provide computationally more efficient ways to represent chirp-like signals compared to the chirplet decompositions.

5. REFERENCES

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