

MULTIWAVELET SYSTEMS WITH DISJOINT MULTISCALING FUNCTIONS

Felix C. A. Fernandes and C. Sidney Burrus

Department of Electrical and Computer Engineering, Rice University, Houston, TX 77251–1892, USA

ABSTRACT

This paper describes the first steps toward a multiwavelet system that may retain the advantages of a traditional multiwavelet system while alleviating some of its disadvantages. We attempt to achieve this through the introduction of a novel property — the disjoint support of the multiscaling functions. We derive the conditions on the matrix filter coefficients that guarantee the disjoint support of multiscaling functions. Our preliminary results demonstrate that multiwavelet systems with this property may be arbitrarily complex. We then establish the existence of multiwavelet systems with two scaling functions and approximation order = 2.

1. INTRODUCTION

Wavelet systems are powerful because they may be designed to possess certain useful properties. Unfortunately, a given wavelet system may not possess all such useful properties simultaneously and we may be forced to choose a subset of the desired properties for a particular application [1]. On the other hand, multiwavelets have attracted considerable attention due to their ability to simultaneously offer a larger subset of useful properties than is possible with wavelet systems. In particular, multiwavelet systems can simultaneously offer real-valued basis functions with compact support, specified approximation order, orthogonality and symmetry [2, 3, 4, 5, 6, 7, 8, 9]. Two disadvantages that accompany multiwavelet systems are the increased computational costs due to the use of multifilters and the urgent need to prefilter data due to spectral imbalance in the multifilter banks [10, 11, 12, 13]. Lebrun et al. [14] and Selesnick [15] circumvent the prefiltering problem by designing “balanced” multiwavelets.

Our motivation for the study of multiwavelet systems whose multiscaling functions have disjoint support (MSMFDS’s) is based on the conjecture that, in a sense, MSMFDS’s may be viewed as being “inbetween” traditional wavelet systems (TWS’s) and traditional multiwavelet systems (TMS’s). We arrived at this conjecture on observing that MSMFDS’s are less restrictive than TWS’s since they employ more than one basis function, but they are more restrictive than TMS’s since the disjoint support property imposes a sparse structure on the matrix filter coefficients of the MSMFDS. This status indicates that MSMFDS’s may be designed to overcome some of the disadvantages of TMS’s while retaining some of their advantages.

In order to characterize MSMFDS’s, we shall start with a TMS, and then determine the conditions to be imposed on its matrix filter coefficients in order to force the support sets of its scaling functions to be disjoint. Thus we arrive at a MSMFDS. We shall then provide an example to demonstrate the existence of an MSMFDS with approximation order = 2.

2. PRELIMINARIES

First, we formally define the support set of a measurable function F from the real $\mathcal{R}$ to the complex $\mathcal{C}$ as done by So et al. in [16]. A point $t \in \mathcal{R}$ is called a *support point* of F if the measure of the intersection $\{x : F(x) \neq 0\} \cap (t - \epsilon, t + \epsilon)$ is not zero for any $\epsilon > 0$. Then the support set of F , denoted by $\text{supp}(F)$ is defined as the convex hull of the set of support points of F .

In a TMS, translates of a scaling vector $\Phi(t) = [\phi_1(t), \dots, \phi_r(t)]^T$ are used to span a subspace of $\mathcal{L}^2(\mathcal{R})$, where r is a positive integer and T refers to a matrix transpose operator. $\phi_1(t), \dots, \phi_r(t)$ are the component multiscaling functions.

The matrix dilation equation

$$\Phi(t) = \sqrt{2} \sum_n C_n \Phi(2t - n) \quad (1)$$

where C_n are $r \times r$ matrices referred to as “matrix filter coefficients”. The matrix dilation equation (1) may be solved in order to obtain the scaling function vector, $\Phi(t)$. We shall view the sequence C_n as an r -input r -output matrix scaling filter for which we adopt the following notation:

$$[C_n]_{i,j} = h_i(nr + j), \quad (2)$$

where $[C_n]_{i,j}$ is the $(i, j)^{\text{th}}$ entry of the matrix C_n . Let us define

$$P(z) = \sum_{k=0}^N C_k z^k \quad (3)$$

to be the *matrix symbol* associated with the matrix scaling coefficients, C_k . $P(z)$ is an $r \times r$ matrix with polynomial entries. Let $[P(z)]_{i,j}$ denote the $(i, j)^{\text{th}}$ polynomial entry of $P(z)$ and $h(i, j)$ (resp., $l(i, j)$) be the highest (resp., lowest) degree of $[P(z)]_{i,j}$. By convention, the highest (resp., lowest) degree of the zero polynomial is $-\infty$ (resp., ∞). Also, $\|P(i, j)\|$ will refer to the length of $[P(z)]_{i,j}$. Evidently, $\|P(i, j)\| = h(i, j) - l(i, j)$.

3. THE DISJOINT SUPPORT CRITERION

Given a scaling vector, $\Phi(t) = [\phi_1(t), \dots, \phi_r(t)]^T$, the disjoint support property of the multiscaling functions requires

$$\text{supp}(\phi_1(t)) \cap \dots \cap \text{supp}(\phi_r(t)) = \emptyset. \quad (4)$$

This work was supported by DARPA, and Texas Instruments.
Email: felixf@rice.edu, csb@rice.edu
Web: http://www.dsp.rice.edu

We shall obtain the disjoint support property by requiring the support sets of the scaling functions to be determined by

$$\text{supp}(\phi_i) = [N_i, N_{i+1}], \quad 1 \leq i \leq r, \quad (5)$$

where $N_i \in \mathcal{R}$, $1 \leq i \leq r+1$.

In order to derive the disjoint support criterion in terms of the matrix filter coefficients, we first relate the degrees of the polynomials in the matrix symbol $P(z)$ as done by So et al. in [16].

For each $1 \leq i \leq r$, using the dilation equation (1), we get

$$\begin{aligned} \phi_i(t) &= \sum_{k=0}^N \sum_{j=1}^r [C_k]_{i,j} \phi_j(2t-k) \\ &= \sum_{j=1}^r \sum_{k=0}^N [C_k]_{i,j} \phi_j(2t-k) \\ &= \sum_{j=1}^r \sum_{k=l(i,j)}^{h(i,j)} [C_k]_{i,j} \phi_j(2t-k). \end{aligned}$$

So et al [16] have shown that:

$$\text{supp}(\phi_i(t)) \subset \text{conv} \left(\bigcup_{j=1}^r \text{supp} \left(\sum_{k=l(i,j)}^{h(i,j)} [C_k]_{i,j} \phi_j(2t-k) \right) \right),$$

where “conv” is the convex hull of a set. Enforcing the disjoint supports of the scaling functions in equation (5) gives us

$$[N_i, N_{i+1}] \subset \text{conv} \left(\bigcup_{j=1}^r \left[\frac{1}{2}(N_j + l(i,j)), \frac{1}{2}(N_{j+1} + h(i,j)) \right] \right).$$

This enables us to deduce that the disjoint support criterion phrased indirectly in terms of the matrix filter coefficients is

$$2N_i = \min_{1 \leq j \leq r} \{N_j + l(i,j)\} \quad (6)$$

$$2N_{i+1} = \max_{1 \leq j \leq r} \{N_{j+1} + h(i,j)\}. \quad (7)$$

4. OBTAINING DISJOINT MULTISCALING FUNCTIONS

The disjoint support criterion in (6, 7) does not explicitly divulge the matrix filter coefficients that are associated with disjoint multiscaling functions – it merely characterizes the highest and lowest degrees of polynomials in the matrix symbol $P(z)$. We also emphasize that the matrix filter coefficients satisfying the disjoint support criterion are not unique, and we need additional constraints to arrive at the matrix filter coefficients satisfying the disjoint support criterion. Additional constraints are readily available in the guise of equations that need to be satisfied for the multiwavelets to enjoy other desirable properties such as a specified approximation order and/or orthogonality.

Unfortunately, the format of the disjoint support criterion (6, 7) renders it difficult to incorporate it into a system of equations whose solution yields the desired matrix filter coefficients. Instead, we may use a heuristic rule to determine the locations of the non-zero entries in the matrix filter coefficients. The system of equations associated with the other desired properties of the multiwavelet system may then be solved using the known structure of the matrix filter coefficients.

TMS's have many evenly distributed non-zero entries in the matrix filter coefficients. Thus, a good choice for the heuristic

rule would be to minimize the number of zero matrix filter coefficient entries by maximizing the lengths of the constituent polynomials in the matrix symbol $P(z)$ and minimizing the variance of the lengths of each of the constituent polynomials (to evenly distribute the non-zero entries). This particular heuristic rule may be expressed mathematically as

$$\text{maximize} \left(\underbrace{\sum_i \sum_j \|P(i,j)\|}_{\text{sum}} \right), \quad (8)$$

$$\text{minimize} \left(\underbrace{\sum_i \sum_j \left(\|P(i,j)\| - \frac{\text{sum}}{r^2} \right)^2}_{\text{variance}} \right). \quad (9)$$

We illustrate the application of this heuristic rule to a system with $r = 2$ in order to obtain the structure of the matrix filter coefficients satisfying the disjoint support criterion (6, 7).

The disjoint support criterion for $r = 2$ with $N_1 = 0$ is

$$0 = \min \{l(1,1), N_2 + l(1,2)\} \quad (10)$$

$$2N_2 = \max \{N_2 + h(1,1), N_3 + h(1,2)\} \quad (11)$$

$$2N_2 = \min \{l(2,1), N_2 + l(2,2)\} \quad (12)$$

$$2N_3 = \max \{N_2 + h(2,1), N_3 + h(2,2)\} \quad (13)$$

It is possible to use an optimization theoretic approach to arrive at the optimal solution to the disjoint support criterion using the heuristic rule. However, since the heuristic is not guaranteed to give an optimal solution, we will be content with a sub-optimal solution obtained in two steps. First, we choose $l(i,j)$ as small as possible and $h(i,j)$ as large as possible, to satisfy the disjoint support criterion (10,11, 12,13). This gives us

$$l(1,1) = 0 \quad (14)$$

$$l(1,2) = 0 \quad (15)$$

$$h(1,1) = N_2 \quad (16)$$

$$h(1,2) = 2N_2 - N_3 \quad (17)$$

$$l(2,1) = 2N_2 \quad (18)$$

$$h(2,1) = 2N_3 - N_2 \quad (19)$$

$$l(2,2) = N_2 \quad (20)$$

$$h(2,2) = N_3. \quad (21)$$

The associated lengths of the constituent polynomials in the matrix symbol $P(z)$ are

$$\|P(1,1)\| = N_2 \quad (22)$$

$$\|P(1,2)\| = 2N_2 - N_3 \quad (23)$$

$$\|P(2,1)\| = 2N_3 - 3N_2 \quad (24)$$

$$\|P(2,2)\| = N_3 - N_2. \quad (25)$$

The second step in the heuristic process involves keeping N_2 constant and determining the value of N_3 that minimizes the variance in (9). The minimizer, N_3 is easily determined to be

$$N_3 = \frac{17N_2}{10} \quad (26)$$

Choosing $N_2 = 10$ and the optimal N_3 from (26), equations (14–21) enable us to determine the forms of the polynomial constituents of the matrix symbol $P(z)$ as

$$[P(z)]_{1,1} = \sum_{k=0}^{10} h_1(2k)z^k \quad (27)$$

$$[P(z)]_{1,2} = \sum_{k=0}^3 h_1(2k+1)z^k \quad (28)$$

$$[P(z)]_{2,1} = \sum_{k=20}^{24} h_2(2k)z^k \quad (29)$$

$$[P(z)]_{2,2} = \sum_{k=10}^{17} h_2(2k+1)z^k. \quad (30)$$

This enables us to determine the structure of the matrix filter coefficients to be

$$C_n = \begin{bmatrix} h_1(2n) & h_1(2n+1) \\ 0 & 0 \end{bmatrix}, \quad 0 \leq n \leq 3 \quad (31)$$

$$= \begin{bmatrix} h_1(2n) & 0 \\ 0 & 0 \end{bmatrix}, \quad 4 \leq n \leq 9 \quad (32)$$

$$= \begin{bmatrix} h_1(2n) & 0 \\ 0 & h_2(2n+1) \end{bmatrix}, \quad n = 10 \quad (33)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & h_2(2n+1) \end{bmatrix}, \quad 11 \leq n \leq 17 \quad (34)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad 18 \leq n \leq 19 \quad (35)$$

$$= \begin{bmatrix} 0 & 0 \\ h_2(2n) & 0 \end{bmatrix}, \quad 20 \leq n \leq 24. \quad (36)$$

It is evident from the structure of the matrix coefficients that the disjoint support property imposes sparsity on the matrix coefficients. We also point out that since N_2 can be made arbitrarily large, we can construct arbitrarily long sequences of matrix coefficients. This is worthy of mention because it demonstrates that the disjoint support property does not restrict the length of the multi-filters and hence it should be possible to impose conditions on the matrix filter coefficients to obtain additional desired properties.

5. AN EXAMPLE: DISJOINT MULTISCALING FUNCTIONS WITH APPROXIMATION ORDER = 2

We now provide an example of a multiwavelet system with disjoint multiscaling functions and approximation order = 2. Consider the sequence of matrix coefficients

$$C_0 = \begin{bmatrix} h_1(0) & h_1(1) \\ 0 & 0 \end{bmatrix}, \quad (37)$$

$$C_1 = \begin{bmatrix} h_1(2) & 0 \\ 0 & h_2(3) \end{bmatrix}, \quad (38)$$

$$C_2 = \begin{bmatrix} 0 & 0 \\ h_2(4) & h_2(5) \end{bmatrix}, \quad (39)$$

$$C_3 = \begin{bmatrix} 0 & 0 \\ h_2(6) & 0 \end{bmatrix}. \quad (40)$$

It is easily verified that this sequence of matrix filter coefficients satisfies the disjoint support criterion (6, 7) with $N_1 = 0$, $N_2 =$

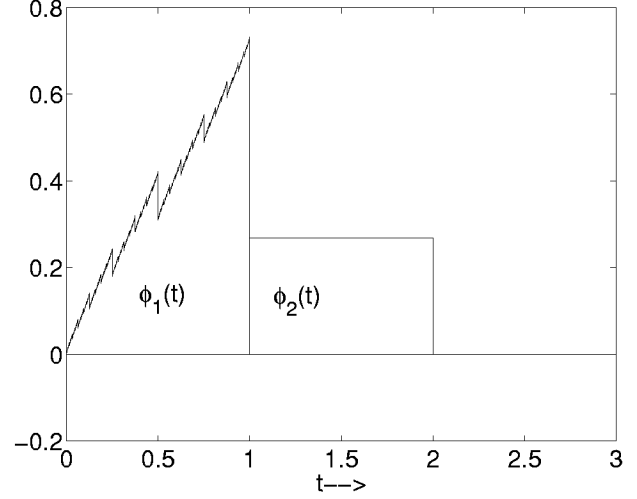


Figure 1: Disjoint multiscaling functions for $r = 2$

1, $N_3 = 2$ and hence will give rise to multiscaling functions with disjoint support. Next, we shall impose additional constraints on this sequence of matrix coefficients to enable it to achieve approximation order = 2.

Strela ([9]) proves that a multi-scaling function $\Phi(t) \in L^1$ with linearly independent translates $\Phi(t - k), k \in \mathbb{Z}$ has approximation order m if and only if there exist (non-zero) vectors $\underline{u}_j, j = 0, \dots, (m-1)$ such that

$$\sum_k \sum_{l=0}^j \binom{j}{l} (-k)^{j-l} \underline{u}_l C(2k+1) = 2^{-j} \underline{u}_j \quad (41)$$

$$\sum_k \sum_{l=0}^j \binom{j}{l} (-k)^{j-l} \underline{u}_l C(2k) = 2^{-j} \sum_{l=0}^j (-1)^{j-l} \binom{j}{l} \underline{u}_l \quad (42)$$

On plugging the matrix coefficients (37–40) arising from the disjoint support criterion in (6, 7), into the equations associated with the approximation order (41, 42), we obtain a set of nonlinear equations which is easily solved using Gröbner bases [17] made available through the software package “Singular” [18]. One of the four resulting solutions is $h_1(0) = 0.5, h_1(1) \neq 0, h_1(2) = 0.5, h_2(3) = 0.5, h_2(4) = 0, h_2(5) = 1; h_2(6) = 0$. When the free parameter $h_1(1)$ is set to 1, the disjoint multiscaling functions shown in figure (1) are obtained. This simple example establishes the existence of multiwavelet systems with disjoint multiscaling functions and specified approximation order.

6. CONCLUSIONS AND FUTURE WORK

We defined the disjoint support property as applicable to the multiscaling functions in a multiwavelet system. Next, we derived the disjoint support criterion relating degrees of polynomials in the matrix symbol of a wavelet system to the boundary points of the support sets of the disjoint multiscaling functions. It is difficult to directly integrate the disjoint support criterion into a larger system of nonlinear equations that injects other desirable properties into a multiwavelet system. This problem was circumvented by applying a heuristic to determine the structure of the matrix coefficients. This structure could then be imposed on a larger system

of nonlinear equations to enable the resulting multiwavelet system to enjoy other desirable properties. Finally, we confirmed the existence of multiwavelet systems with disjoint multiscaling functions and approximation order = 2.

Our preliminary results are encouraging since

1. they illustrate that the disjoint support property is not overly restrictive. In particular, we established that although multifilters with the disjoint support property are sparse, they may be arbitrarily long. This is significant since short, sparse multifilters would curtail the available degrees of freedom making it difficult for the multiwavelet system to enjoy other desirable properties.
2. they confirm the existence of multiwavelet systems with the disjoint support property and specified approximation order, in a short multifilter.

Further work remains to be done in order to determine the applicability of MSMFDS's to practical problems. The consequences of enforcing disjoint support of the multiwavelet functions (as opposed to disjoint support of the multiscaling functions) may also prove useful in this regard. The time-domain techniques for the design of multiwavelet systems with the disjoint support property presented here are inconvenient for the design of complex systems due to the need to solve large sets of nonlinear equations. It would thus be profitable to investigate frequency-domain design techniques.

To conclude, we reiterate our conjecture that MSMFDS's are "inbetween" traditional wavelet systems and traditional multiwavelet systems. They have more degrees of freedom than traditional wavelet systems since they exploit several basis functions, yet they are a bit more restrictive than traditional multiwavelet systems due to the sparsity of their matrix coefficients. This observation justifies further work channelled toward reaping the benefits of multiwavelet systems, while being able to avoid some of their inherent disadvantages, such as the dire necessity to prefilter, and increased computational complexity.

7. ACKNOWLEDGEMENTS

The authors thank Ivan Selesnick for his assistance with "Singular" and for promptly providing us with other software.

8. REFERENCES

- [1] I. Daubechies, *Ten Lectures on Wavelets*. Philadelphia, PA: SIAM, 1992. Notes from the 1990 CBMS-NSF Conference on Wavelets and Applications at Lowell, MA.
- [2] C. K. Chui and J. Lian, "A study of orthonormal multi-wavelets," *Applied Numerical Mathematics*, vol. 20, pp. 273–298, Mar. 1996.
- [3] T. Cooklev, A. Nishihara, M. Kato, and M. Sablatash, "Two-channel multifilter banks and multiwavelets," in *IEEE Proc. Int. Conf. Acoust., Speech, Signal Processing*, vol. 5, (Atlanta), pp. 2769–2772, May 1996.
- [4] G. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust, "Construction of orthogonal wavelets using fractal interpolation functions," *SIAM Journal of Mathematical Analysis*, vol. 27, pp. 1158–1192, July 1996.
- [5] J. S. Geronimo, D. P. Hardin, and P. R. Massopust, "Fractal functions and wavelet expansions based on several scaling functions," *Journal of Approximation Theory*, vol. 78, pp. 373–401, Sept. 1994.
- [6] P. Rieder, J. Gotze, and J. A. Nossek, "Algebraic design of discrete multiwavelet transforms," in *IEEE Proc. Int. Conf. Acoust., Speech, Signal Processing*, vol. 3, pp. 17–20, 1994.
- [7] P. Rieder, "Parameterization of symmetric multiwavelets," in *IEEE Proc. Int. Conf. Acoust., Speech, Signal Processing*, vol. 3, pp. 2461–2464, Apr. 1997.
- [8] G. Strang and V. Strela, "Orthogonal multiwavelets with vanishing moments," *J. Optical Eng.*, vol. 33, pp. 2104–2107, 1994.
- [9] V. Strela, *Multiwavelets: Theory and Applications*. PhD thesis, M.I.T., June 1996.
- [10] D. P. Hardin and D. W. Roach, "Multiwavelet prefilters i: Orthogonal prefilters preserving approximation order $p \leq 2$." Preprint, 1997.
- [11] M. J. Vrhel and A. Aldroubi, "Pre-filtering for the initialization of multi-wavelet transforms," in *IEEE Proc. Int. Conf. Acoust., Speech, Signal Processing*, vol. 3, (Munich), pp. 2033–2036, Apr. 1997.
- [12] X. Xia, J. S. Geronimo, D. P. Hardin, and B. W. Suter, "Design of prefilters for discrete multiwavelet transforms," *IEEE Transactions on Signal Processing*, vol. 44, pp. 25–35, Jan. 1996.
- [13] X. Xia, "A new prefilter design for discrete multiwavelet transforms," *IEEE Transactions on Signal Processing*, vol. 46, pp. 1558–1570, June 1998.
- [14] J. Lebrun and M. Vetterli, "High order balanced multiwavelets," in *IEEE Proc. Int. Conf. Acoust., Speech, Signal Processing*, (Seattle, WA), 1998.
- [15] I. Selesnick, "Multiwavelet bases with extra approximation properties," Aug 1997.
- [16] W. So and J. Wang, "Estimating the support of a scaling vector," *SIAM Journal of Matrix Analysis Applications*, vol. 18, pp. 66–73, Jan. 1997.
- [17] R. Fröberg, *An Introduction to Gröbner Bases*. John Wiley and Sons, 1997.
- [18] G. M. Greuel, G. Pfister, and H. Schönemann, *Singular Reference Manual*. Centre for Computer Algebra, University of Kaiserslautern, May 1997. <http://www.mathematik.uni-kl.de/zca/Singular>.