

SAMPLING THEOREMS FOR LINEAR TIME-VARYING SYSTEMS WITH BANDLIMITED INPUTS

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ABSTRACT

We propose and prove an extended sampling theorem for linear time-varying systems. As a result, we establish a discrete-time equivalent model of the input-output relation of the system for the case of bandlimited inputs and bandlimited system variation. The sampling of the output signal and an equivalent discrete-time model of the system are discussed.

1. INTRODUCTION

The famous Shannon Sampling Theorem has long been an interesting object for theorists in the areas of mathematics and signal theory to review and extend repeatedly. Even though the root of this theorem may be traced back to several pioneers in the early 19th century as mentioned in [1], a lot of related work or various extensions of this theorem had been done along the several decades up to the late 1970's.

In view of the class of the signals, the original sampling theorem for a deterministic bandlimited signal was extended to the sampling of wide-sense stationary stochastic processes [2], [3] and then even further to nonstationary random processes [4], [5], [6].

However, we have not found many published results related to sampling considerations for input-output relationships of stochastic linear time-varying systems. In [7] sampling itself is considered as a process involving a linear time-varying system, but this is not our goal. Rather, we seek to understand the conditions under which a continuous-time model can be replaced with an equivalent discrete-time model with no loss of information at the input or output.

Recent trends in signal processing show that discrete-time signal models become important to use the computing power of digital systems effectively. Thus, we here extend the sampling theorem to the input-output relation of linear time-varying (LTV) systems when the input is bandlimited.

2. EXTENSION OF THE SAMPLING THEOREM FOR LTV SYSTEMS: DETERMINISTIC CASE

Let $x(t)$ and $y(t)$ be the input and output signals of a LTV system having its impulse response $g(t, \alpha)$ which is the output of the system at time t when an impulse is applied to the system at time $t - \alpha$. Then $y(t)$ can be represented as

$$y(t) = \int_{-\infty}^{\infty} x(t - \alpha)g(t, \alpha) d\alpha. \quad (1)$$

Applying a 2-dimensional Fourier transform to (1), we have

$$Y(\nu) = \int_{-\infty}^{\infty} X(f)G(f - \nu, f) df \quad (2)$$

where $X(f)$ and $Y(\nu)$ are the Fourier transforms of $x(t)$ and $y(t)$, respectively, and

$$G(\nu, f) \triangleq \int \int g(t, \alpha) e^{-j2\pi\nu t} e^{-j2\pi f \alpha} dt d\alpha. \quad (3)$$

We can see from this equation that the spectrum of the system output is the superposition of the product of $X(f)$ and the differential frequency gain $G(f - \nu, f) df$, which is a diagonal slice of the 2-D spectrum as shown in Fig. 1, along the f -axis.

In the figure, B_i and B_s represent the bandwidths of $x(t)$ and the system variation, respectively.

Let B_o be the bandwidth of the output signal $y(t)$. Then it can be easily seen that the possible value of B_o resides between $\max\{B_i, B_s\}$ and $B_i + B_s$.

Therefore, if we choose a common interval T for the sampling of $x(t)$ and $y(t)$ such that $T \leq \frac{1}{2B_o}$, then

$$y(kT) = \sum_{n=-\infty}^{\infty} x(kT - nT)\tilde{g}(kT, nT) \quad (4)$$

where

$$\tilde{g}(kT, nT) \triangleq T \int_{-B_o}^{B_o} \int_{-B_o}^{B_o} G(\nu, f) e^{j2\pi(fnT + \nu kT)} df d\nu.$$

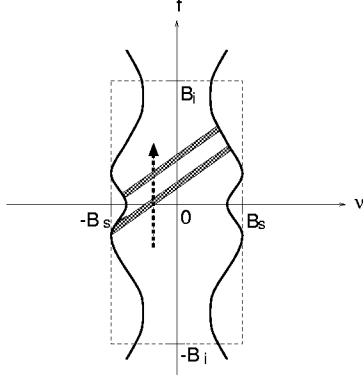


Figure 1: Top view of $G(\nu, f)$.

3. EXTENSION OF THE SAMPLING THEOREM FOR LTV SYSTEMS: STOCHASTIC CASES

Sampling may be considered for the cases when the input signal and/or the system impulse response are stochastic processes.

Let $\varphi(t; B)$ denote a *sinc* function with the bandwidth B . The following is our definition for bandlimit-ness of a stochastic signal.

Definition A stochastic signal $s(t)$, satisfying $E\{s^2(t)\} < \infty$, all t , is said to be *mean-square (MS) B -bandlimited* if the relation

$$s(t) \stackrel{\text{m.s.}}{=} \int_{-\infty}^{\infty} s(\tau) \varphi(t - \tau; B) d\tau \quad (5)$$

holds¹.

Similarly, a 2-dimensional stochastic signal $s(t, u)$, satisfying $E\{s^2(t, u)\} < \infty$, all t and u , is said to be *MS (B_1, B_2) -bandlimited* if the relations

$$s(t, u) \stackrel{\text{m.s.}}{=} \int_{-\infty}^{\infty} s(\tau, u) \varphi(t - \tau; B_1) d\tau \quad (6)$$

and

$$s(t, u) \stackrel{\text{m.s.}}{=} \int_{-\infty}^{\infty} s(t, \sigma) \varphi(u - \sigma; B_2) d\sigma \quad (7)$$

hold.

Note that these MS bandlimited stochastic processes do not need to be stationary.

Then, we have some useful relations for MS B -bandlimited stochastic processes as follows.

Relation Let $s(t)$ be a MS B -bandlimited stochastic process. Then, for its correlation function $R_{ss}(t, u) \triangleq$

¹ "m.s." denotes "equality in the mean-square sense".

$E\{s(t)s^\dagger(u)\}$, the following equalities hold:

$$R_{ss}(t, u) = \int_{-\infty}^{\infty} R_{ss}(t, \sigma) \varphi(u - \sigma; B) d\sigma \quad (8)$$

$$= \int_{-\infty}^{\infty} R_{ss}(\tau, u) \varphi(t - \tau; B) d\tau \quad (9)$$

$$= \sum_{k=-\infty}^{\infty} R_{ss}(k, u) \varphi(t - k; B) \quad (10)$$

$$= \sum_{l=-\infty}^{\infty} R_{ss}(t, l) \varphi(u - l; B). \quad (11)$$

Note that we can also write

$$R_{ss}(t, u) = \iint R_{ss}(\tau, \sigma) \varphi(u - \sigma; B) \varphi(t - \tau; B) d\sigma d\tau$$

or,

$$R_{ss}(t, u) = \sum_k \sum_l R_{ss}(k, l) \varphi(t - k; B) \varphi(u - l; B).$$

Now, let $x(t)$ be MS B_i -bandlimited and $g(t, \alpha)$ MS B_s -bandlimited in the t -variable. Then $y(t)$ is MS bandlimited to B_o where $B_o = B_i + B_s$ and we have the following theorem through the help of the above relation.

Theorem (Sampling Theorem for LTV Systems)

Let $x(t)$ be MS B_i -bandlimited and $g(t, \alpha)$ MS B_s -bandlimited in the t -variable. Then $y(t)$ is MS B_o -bandlimited, where $B_o = B_i + B_s$, and we have

$$y(t) \stackrel{\text{m.s.}}{=} \sum_{k=-\infty}^{\infty} \varphi(t - kT; B_o) \sum_{n=-\infty}^{\infty} x(kT - nT) \tilde{g}(kT, nT) \quad (12)$$

where $\tilde{g}(t, \alpha)$ is the projection of $g(t, \alpha)$ onto the space of 2-D stochastic processes which are MS B_i -bandlimited in the α variable, and $T \leq \frac{1}{2B_o}$.

In the theorem, we can see that the equivalent discrete-time model of a LTV system for bandlimited inputs may be represented as the series of 2-dimensional samples of the projection of the original process $g(t, \alpha)$ onto MS bandlimited signal space, with the sampling rate of T .

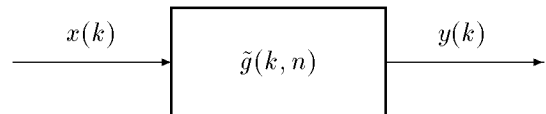


Figure 2: Equivalent discrete model of a LTV system.

Therefore, we have an equivalent discrete description of LTV systems as shown in Fig. 2 in which $\tilde{g}(k, n)$

represents the sampled-version of $\tilde{g}(t, \alpha)$ with respect to the two variables, t and α , at the rate of $\frac{1}{2B_o}$.

There is an analogous result for deterministic systems with stochastic inputs which is omitted here for brevity.

4. CONCLUSIONS

An extension of the sampling theorem for the input-output relation of LTV systems was considered when its input is bandlimited.

This sampling theorem may be applied to the modeling of wireless channels which are generally represented as LTV systems [8].

5. REFERENCES

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Appendix: Proof of the theorem

Proof. WOLOG we let $T = 1$ and omit the domain of integration which is the set of real numbers, and the

domain of summation which is the set of integers, just for convenience.

Now let

$$\hat{y}(t) = \sum_k \varphi(t - kT; B_o) \sum_n x(kT - nT) \tilde{g}(kT, nT).$$

Then

$$\begin{aligned} E\{|y(t) - \hat{y}(t)|^2\} \\ = R_{yy}(t, t) - R_{\hat{y}y}(t, t) - R_{y\hat{y}}(t, t) + R_{\hat{y}\hat{y}}(t, t) \\ = 0, \end{aligned}$$

since

$$\begin{aligned} R_{yy}(t, t) &\triangleq E\{y(t)y^\dagger(t)\} \\ &= E \int x(t - \alpha)g(t, \alpha) d\alpha \int x(t - \beta)g(t, \beta) d\beta \\ &= E \int \left[\int x(t - \omega)\varphi(-\alpha + \omega; B_i) d\omega \right] \\ &\quad \cdot g(t, \alpha) d\alpha \int [x(t - \nu)\varphi(-\beta + \nu; B_i) d\nu] \\ &\quad \cdot g(t, \beta) d\beta \\ &= E \iint x(t - \omega)x(t - \nu) \\ &\quad \cdot \int g(t, \alpha)\varphi(-\alpha + \omega; B_i) d\alpha \\ &\quad \cdot \int g(t, \beta)\varphi(-\beta + \nu; B_i) d\beta d\omega d\nu \\ &= \iint R_{xx}(t - \omega, t - \nu) R_{\tilde{g}\tilde{g}}(t, t; \omega, \nu) d\omega d\nu, \end{aligned}$$

$$\begin{aligned} R_{\hat{y}\hat{y}}(t, t) &\triangleq E\{\hat{y}(t)\hat{y}^\dagger(t)\} \\ &= E \int x(t - \alpha)g(t, \alpha) d\alpha \sum_k \varphi^\dagger(t - k; B_o) \\ &\quad \cdot \sum_n x^\dagger(k - n)\tilde{g}^\dagger(k, n) \\ &= E \int x(\alpha)g(t, t - \alpha) d\alpha \\ &\quad \cdot \sum_k \varphi(t - k; B_o) \sum_n x^\dagger(n)\tilde{g}^\dagger(k, k - n) \end{aligned} \quad (13)$$

$$\begin{aligned} &= E \int \left[\int x(\omega)\varphi(\alpha - \omega; B_i) d\omega \right] g(t, t - \alpha) d\alpha \\ &\quad \cdot \sum_k \varphi(t - k; B_o) \sum_n x^\dagger(n)\tilde{g}^\dagger(k, k - n) \end{aligned} \quad (14)$$

$$\begin{aligned} &= E \int x(\omega) \left[\int g(t, t - \alpha)\varphi(\alpha - \omega; B_i) d\alpha \right] d\omega \\ &\quad \cdot \sum_k \varphi(t - k; B_o) \sum_n x^\dagger(n)\tilde{g}^\dagger(k, k - n) \end{aligned} \quad (15)$$

$$(16)$$

$$= \int \sum_n R_{xx}(\omega, n) \sum_k R_{\tilde{g}\tilde{g}}(t, k; t - \omega, k - n) \cdot \varphi(t - k; B_o) d\omega \quad (17)$$

$$= \int \int R_{xx}(\omega, \beta) \sum_n R_{\tilde{g}\tilde{g}}(t, t; t - \omega, t - n) \cdot \varphi(n - \beta; B_i) d\omega d\beta \quad (18)$$

$$= \int \int R_{xx}(\omega, \beta) R_{\tilde{g}\tilde{g}}(t, t; t - \omega, t - \beta) d\omega d\beta \quad (19)$$

$$= \int \int R_{xx}(t - \omega, t - \beta) R_{\tilde{g}\tilde{g}}(t, t; \omega, \beta) d\omega d\beta, \quad (20)$$

$$R_{\tilde{g}\tilde{g}}(t, t) \triangleq \int \int R_{xx}(t - \beta, t - \omega) R_{\tilde{g}\tilde{g}}(t, t; \beta, \omega) d\beta d\omega, \quad (21)$$

$$R_{\tilde{g}\tilde{g}}(t, t) \triangleq \sum_k \varphi(t - k; B_o) \sum_l \varphi^\dagger(t - l; B_o) \cdot \sum_n \sum_m x(k - n) x^\dagger(l - m) \tilde{g}(k, n) \tilde{g}^\dagger(l, m) \quad (22)$$

$$= \sum_k \varphi(t - k; B_o) \sum_l \varphi(t - l; B_o) \cdot \sum_n \sum_m x(n) x^\dagger(m) \tilde{g}(k, k - n) \tilde{g}^\dagger(l, l - m) \quad (23)$$

$$= \sum_n \sum_m R_{xx}(n, m) \sum_k \sum_l R_{\tilde{g}\tilde{g}}(k, l; k - n, l - m) \cdot \varphi(t - k; B_o) \varphi(t - l; B_o) \quad (24)$$

$$= \int \int R_{\tilde{g}\tilde{g}}(t, t; t - \alpha, t - \beta) \sum_n \sum_m R_{xx}(n, m) \cdot \varphi(-n + \alpha; B_i) \varphi(-m + \beta; B_i) d\alpha d\beta \quad (25)$$

$$= \int \int R_{xx}(\alpha, \beta) R_{\tilde{g}\tilde{g}}(t, t; t - \alpha, t - \beta) d\alpha d\beta \quad (26)$$

$$= \int \int R_{xx}(t - \alpha, t - \beta) R_{\tilde{g}\tilde{g}}(t, t; \alpha, \beta) d\alpha d\beta. \quad (27)$$

In the steps from (17) to (18)

$$R_{xx}(\alpha, n) = \int R_{xx}(\alpha, \beta) \varphi(n - \beta; B_i) d\beta \quad (28)$$

and

$$\sum_k R_{\tilde{g}\tilde{g}}(t, k; t - \alpha, k - n) \varphi(t - k) = R_{\tilde{g}\tilde{g}}(t, t; t - \alpha, t - n)$$

were applied.

In the steps from (18) to (19)

$$\sum_n R_{\tilde{g}\tilde{g}}(t, t; t - \alpha, t - n) \varphi(n - \beta) = R_{\tilde{g}\tilde{g}}(t, t; t - \alpha, t - \beta) \quad (29)$$

was used.

Note that $\tilde{g}(\cdot, \cdot)$ is the 2-D inverse Fourier transform of a B_i -truncated version in frequency domain of $g(\cdot, \cdot)$ with respect to the variable α .

(21) can be derived similarly as in the derivation of $R_{yy}(t, t)$.

In the step from (22) to (23), we used the relation

$$\sum_n \sum_m x(k - n) x^\dagger(l - m) \tilde{g}(k, n) \tilde{g}^\dagger(l, m) = \sum_n \sum_m x(n) x^\dagger(m) \tilde{g}(k, k - n) \tilde{g}^\dagger(l, l - m).$$

In the step from (24) to (25), the relations,

$$\sum_k \sum_l R_{\tilde{g}\tilde{g}}((k, l; k - n, l - m) \varphi(t - k) \varphi(t - l) \quad (30)$$

$$= R_{\tilde{g}\tilde{g}}(t, t; t - n, t - m) \quad (31)$$

$$= \int R_{\tilde{g}\tilde{g}}(t, t; t - \alpha, t - m) \varphi(-n + \alpha; B_i) d\alpha \quad (32)$$

$$= \int \int R_{\tilde{g}\tilde{g}}(t, t; t - \alpha, t - \beta) \varphi(-m + \beta; B_i) \cdot \varphi(-n + \alpha; B_i) d\beta d\alpha$$

were used.

In the steps from (25) to (27), the relation

$$\sum_n \sum_m R_{xx}(n, m) \varphi(-n + \alpha; B_i) \varphi(-m + \beta; B_i) = R_{xx}(\alpha, \beta) \quad (33)$$

was used along with a simple change of variables.

Therefore, the sampled sequence $y(k)$ of the stochastic system output where the input is a bandlimited stochastic process, may be represented as

$$y(k) = \sum_{n=-\infty}^{\infty} x(k - n) \tilde{g}(k, n) \quad (34)$$

in the mean square sense. $\square$