

# AN INVERSE SIGNAL APPROACH TO COMPUTING THE ENVELOPE OF A REAL VALUED SIGNAL

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## ABSTRACT

We address the problem of estimating the envelope of a real-valued signal,  $s(t)$ , that is observed for a duration of  $T$  seconds. We model  $s(t)$  using a Fourier series, by considering periodic extensions of the signal. By using an analog of the autocorrelation method of linear prediction on the Fourier coefficients of  $s(t)$ , the envelope of the signal is estimated without explicitly computing the analytic signal through Hilbert transformation. Using this method the envelope of a non-stationary signal can be computed by processing the signal through a sliding  $T$ -second window.

## 1. INTRODUCTION

Instantaneous attributes of a signal, namely its envelope and instantaneous frequency (IF) are often used to characterize a band-pass signal [1]. They are uniquely defined for analytic signals [2]. The analytic signal [2, 3] corresponding to a real signal,  $s(t)$ , is defined as  $s(t) \pm j\hat{s}(t)$  where  $\hat{s}(t)$  is the Hilbert transform of  $s(t)$ , i.e.  $\hat{s}(t) = 1/\pi \int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau} d\tau$  [2]. If we rewrite  $s(t) \pm j\hat{s}(t)$  as  $a(t)e^{\pm j\phi(t)}$ , then  $a(t)$  is the envelope and  $\pm \frac{1}{2\pi} \frac{d\phi(t)}{dt}$  is the IF. The spectrum of an analytic signal vanishes for either positive or negative frequencies. These special signals play fundamental roles in many applications such as narrowband communications and band-pass sampling [4].

To form the analytic signal one has to compute the Hilbert transform of the real-valued signal  $s(t)$ . This leads to a number of difficulties. For instance, the signal  $s(t)$ , in practice, is known only for a finite, and often a short duration. Hence, any physically realizable Hilbert transformer will introduce significant transients in  $\hat{s}(t)$ . This alters the formed analytic signal's characteristics. Thus, if possible, it is desirable to avoid forming the analytic signal before one computes the envelope or the IF.

In this paper, we propose an approach to estimating the envelope of a real-valued signal without explicitly computing its Hilbert transform. We exploit the analogy between the spectral envelope of a sequence and the envelope of a signal. We borrow a well known method from spectral analysis used for modeling the spectral envelope of a sequence and adapt it for modeling the envelope of a signal. The proposed method is an extension of the Linear Prediction in Spectral Domain (LPSD) algorithm that we proposed recently [5–7].

## 2. MODELING THE SPECTRAL ENVELOPE OF A SYMMETRIC SEQUENCE

Consider a sequence  $x(n)$  consisting of  $2N + 1$  complex valued samples denoted by  $x(-N), \dots, x(0), \dots, x(N)$ .  $X(z) = \sum_{n=-N}^N x(n)z^{-n}$ . Let  $b(1), b(2), \dots, b(L)$  represent the coefficients of an FIR filter with system function  $B(z)$ . Let  $F(z) = X(z)B(z)$ .  $F(z) = \sum_{n=-N}^{N+L} f(n)z^{-n}$ . In the autocorrelation method of linear prediction [8, 9], the filter coefficients  $b(1)$  to  $b(L)$  are determined by minimizing the prediction error given by

$$\sum_{n=-N}^{N+L} |f(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{j\omega})|^2 d\omega \quad (1)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})B(e^{j\omega})|^2 d\omega \quad (2)$$

The above error is minimized by finding the  $b(l)$  that satisfy the following simultaneous linear equations

$$\sum_{l=1}^L b(l)r(i-l) = -r(i) \quad 1 \leq i \leq L, \quad (3)$$

where  $r(i) = \sum_{n=-N}^{N-|i|} x^*(n)x(n+i)$  and  $r(i) = r^*(-i)$ . Minimizing the prediction error given in Eq.(1) amounts to flattening the spectral envelope of the sequence  $f(n)$ . Hence,  $1/|B(e^{j\omega})|$  provides a smooth fit to the envelope of the spectrum of  $x(n)$ . The closeness of the fit depends on the order of the filter  $B(z)$ . It is well known that  $B(z)$  is a minimum-phase filter [8, 9].  $B(z)$  is known as the Inverse Filter or Prediction-Error Filter. The above method is often used in speech processing (see figure 3 in Ref.[9], for example) to model the spectral envelope of speech signals over contiguous blocks of data. We wish to adapt this method to obtain the envelope of a real-valued signal.

Let us assume that the sequence,  $x(n)$ , has conjugate symmetry. That is  $x(n) = x^*(-n)$ . Thus  $X(e^{j\omega})$  is real-valued. Let us split the sequence  $x(n)$  into three parts  $x_1(n)$ ,  $x_2(n)$  and  $x_3(n)$ .  $x_1(n)$  corresponds to the samples  $x(-N), x(-N+1), \dots, x(-K)$ ,  $x_2(n)$  corresponds to  $x(-K+1), x(-K+2), \dots, x(0), \dots, x(K-2), x(K-1)$ , and  $x_3(n)$  corresponds to the samples  $x(K), x(K+1), \dots, x(N)$ .  $K$  is some positive integer less than  $N$ . Clearly,  $x(n) = x_1(n) + x_2(n) + x_3(n)$ . Note that  $x_1(n) = x_3^*(-n)$ . Using the above notation we may write the prediction error as follows.

$$f(n) = x(n) * b(n) = \sum_{k=1}^3 x_k(n) * b(n) \quad (4)$$

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\* denotes linear convolution. Let us further assume that the sequence  $x_2(n)$  is identically zero. Then  $f(n) = x_1(n) * b(n) + x_3(n) * b(n)$ . If  $2K - 1 \geq L$ , then the results of the two convolutions  $f_1(n) = x_1(n) * b(n)$  and  $f_3(n) = x_3(n) * b(n)$  do not overlap. Hence the prediction error may be written as follows.

$$\begin{aligned} \sum_{n=-N}^{N+L} |f(n)|^2 &= \sum_{n=-N}^{-K+L} |f_1(n)|^2 + \sum_{n=K}^{N+L} |f_3(n)|^2 \quad (5) \\ &= 1/2\pi \int_{-\pi}^{\pi} |X_1(e^{j\omega})B(e^{j\omega})|^2 d\omega \quad (6) \\ &\quad + 1/2\pi \int_{-\pi}^{\pi} |X_3(e^{j\omega})B(e^{j\omega})|^2 d\omega \end{aligned}$$

Since  $x_1(n) = x_3^*(-n)$ , it follows that  $X_1(e^{j\omega}) = X_3^*(e^{j\omega})$ . Hence, the two terms in the above are identical. This is true even if  $N = \infty$ . Therefore, the inverse filter coefficients and consequently the spectral envelope may be obtained by minimizing either one of the two terms. This implies that, for a sequence with conjugate symmetry or antisymmetry, with sufficient number of zeros in the middle of the sequence, we can obtain a fit to the spectral envelope from either the causal or the anticausal part of the sequence. This observation, although of not much value in this domain, is very useful in obtaining the estimate of the envelope and IF of a signal without computing its analytic version. We shall demonstrate this in the next section.

### 3. ENVELOPE OF REAL-VALUED SIGNAL

Consider a real-valued signal,  $s(t)$ , over an interval of 0 to  $T$  seconds.  $s(t)$  may be represented over this interval by  $2N + 1$  Fourier coefficients for sufficiently large  $N$ . That is

$$s(t) = \sum_{k=-N}^N a_k e^{jk\Omega t}. \quad (7)$$

$\Omega = 2\pi/T$ . Since  $s(t)$  is real-valued  $a_{-k} = a_k^*$ . Let  $s(t)$  be such that the Fourier coefficients  $a_{-K+1}, a_0, \dots, a_{K-1}$  are equal to zero for some  $K < N$ . That is,  $s(t)$  is a band-pass signal. Clearly, the Fourier coefficients,  $a_k$ , are analogous to the conjugate-symmetric sequence,  $x(n)$ , considered in the previous section. Extending the analogy further, let us define an error signal  $e(t)$  over 0 to  $T$  seconds as follows.

$$e(t) = s(t)h(t), \quad (8)$$

where  $h(t) = 1 + \sum_{l=1}^L h_l e^{jl\Omega t}$ . Analogous to the inverse filter  $B(z)$ , defined in the previous section, we may call  $h(t)$  as the 'Inverse Signal'. The goal is to find an inverse signal  $h(t)$  such that the envelope of the error signal  $e(t)$  is flattened. With this in mind, we shall minimize the energy in the error signal  $e(t)$ , i.e.,  $\int_0^T |e(t)|^2 dt$  by choosing the coefficients,  $h_l$ . Plugging in the expression for  $s(t)$  from Eq.(7) into the error-energy expression we get

$$\int_0^T |e(t)|^2 dt = \int_0^T |s(t)h(t)|^2 dt \quad (9)$$

$$= T \sum_{n=-N}^{N+L} |g_n|^2 \quad (10)$$

where  $g_n = a_n * h_n$ .  $h_0 = 1$ . The similarity between the above expression and Eq.(1) is obvious. Again, the inverse signal coefficients,  $h_l$ , can be determined by solving the linear equations analogous to the one in Eq.(3), where  $b(l)$  is replaced by  $h_l$ , and  $a_n$  is substituted for  $x(n)$ . Analogous to the spectral envelope in the previous section, the signal envelope is given by  $1/|h(t)|$ .

Since the Fourier coefficients  $a_{-K+1}$  to  $a_{K-1}$  are all assumed to be zero, the expression for the error-energy can be written as a sum of two terms, if  $L \leq 2K - 1$ .

$$\int_0^T |e(t)|^2 dt = T \sum_{n=-N}^{-K+L} |g_n|^2 + T \sum_{n=K}^{N+L} |g_n|^2 \quad (11)$$

$$= \int_0^T |(s(t) - j\hat{s}(t))h(t)|^2 dt \quad (12)$$

$$+ \int_0^T |(s(t) + j\hat{s}(t))h(t)|^2 dt \quad (13)$$

Since the analytic  $(s(t) + j\hat{s}(t))$  and antianalytic  $(s(t) - j\hat{s}(t))$  signals are complex conjugates of each other, the two terms in the above expression are equal. Thus the inverse signal obtained by minimizing any one of the terms in the above expression is equal to the  $h(t)$  obtained by minimizing the error in Eq.(9) (using the real-valued  $s(t)$ ). Thus  $1/|h(t)|$  gives the Hilbert envelope of the analytic (also antianalytic) signal, although it is computed directly from the real-valued  $s(t)$ . However, for this to be true the real-valued signal  $s(t)$  must have sufficient number of zero-valued Fourier coefficients in the low frequency region.

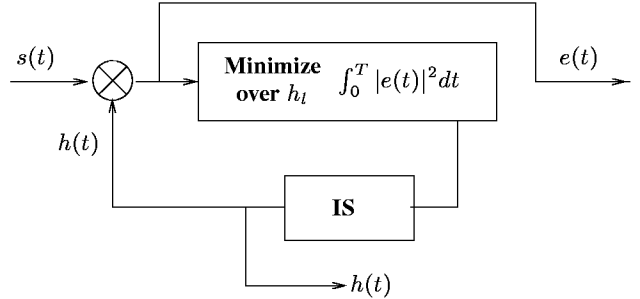


Figure 1: **Envelope Estimator:**  $h(t) = 1 + \sum_{l=1}^L h_l e^{jl\Omega t}$ .  $1/|h(t)|$  gives the estimate of the envelope.  $s(t)$  is a real-valued signal. **IS** stands for Inverse Signal generator.

In the above, since linear prediction is performed on the Fourier coefficients of the signal  $s(t)$ , instead of the signal samples, we called this method, Linear Prediction in the Spectral Domain or LPSD [6]. If we denote  $s(t) + j\hat{s}(t)$  by  $a(t)e^{j\phi(t)}$ , clearly  $|h(t)| \approx \frac{1}{a(t)} = e^{-\ln(a(t))}$ . Further, since  $h(t)$  is a minimum-phase signal [10], its log-envelope and phase are related by Hilbert transform. Therefore  $h(t)$  must be such that

$$h(t) \approx e^{-\ln a(t) - j\widehat{\ln a(t)}}. \quad (14)$$

Recall that the hat denotes the Hilbert transform. Thus the error signal  $e(t)$  may be written as

$$e(t) = s(t)h(t) \approx \frac{1}{2}e^{j(\phi(t) - \widehat{\ln a(t)})} + \frac{1}{2}e^{j(-\phi(t) - \widehat{\ln a(t)})} \quad (15)$$

It can be shown that  $\frac{d(\phi(t) - \widehat{\ln a(t)})}{dt}$  is always positive [6]. It is called the positive instantaneous frequency (PIF) of  $s(t)$ .

#### 4. SIMULATION RESULTS

In this section we present an algorithm for computing the estimate of the envelope of the real signal  $s(t)$ . The algorithm amounts to performing linear prediction on the discrete Fourier transform (DFT) values of the signal samples. Let  $s(n)$  ( $n = 0, 1, \dots, Q$ ), denote the samples of  $s(t)$ . Let  $\Omega = \frac{2\pi}{Q+1}$  be the assumed fundamental frequency. By replacing  $h(t)$  and  $e(t)$  in Eq. 8 by their respective sampled versions, we have

$$e(n) = s(n) + \sum_{l=1}^L h_l s(n) e^{j l \Omega n}, \quad (16)$$

which can be further expressed in matrix notation as

$$\begin{pmatrix} s(0) & s(0) & \cdots & s(0) \\ e^{j\Omega} s(1) & e^{j2\Omega} s(1) & \cdots & e^{jL\Omega} s(1) \\ \vdots & \vdots & \ddots & \vdots \\ e^{jQ\Omega} s(Q) & e^{j2Q\Omega} s(Q) & \cdots & e^{jLQ\Omega} s(Q) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{pmatrix} + \begin{pmatrix} s(0) \\ s(1) \\ \vdots \\ s(Q) \end{pmatrix} = \begin{pmatrix} e(0) \\ e(1) \\ \vdots \\ e(Q) \end{pmatrix}. \quad (17)$$

If we let  $\mathbf{s}$ ,  $\mathbf{H}$ ,  $\mathbf{h}$  and  $\mathbf{e}$  denote the vectors/matrices from left to right in Eq. 17, then the solution vector,  $\mathbf{h}$ , that minimizes  $\mathbf{e}^\dagger \mathbf{e} = \sum_{n=0}^Q |e(n)|^2$  is given by  $\mathbf{h} = -(\mathbf{H}^\dagger \mathbf{H})^{-1} \mathbf{H}^\dagger \mathbf{s}$ .  $\dagger$  stands

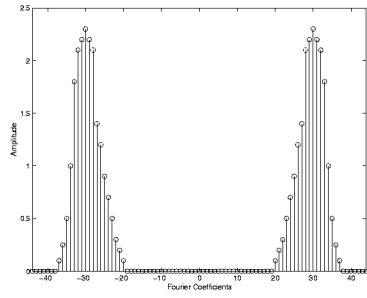
for conjugate-transpose. The solution depends only on the magnitude of  $s[n]$ .  $h[n]$  is reconstructed by substituting the elements of  $\mathbf{h}$  in  $h(n) = 1 + \sum_{l=1}^L h_l e^{j l \Omega n}$ . The envelope estimate is  $\frac{1}{|h(n)|}$ . Some simulation results are provided in Figure 2. A real signal  $s(n)$  shown in Fig.2(b) was synthesized using 36 Fourier coefficients whose magnitudes are shown in Fig.2(a). Their phase angles were chosen randomly. The envelope of the signal was estimated using the above algorithm for two different values of  $L$ . Fig.2(c) and Fig.2(d) display the envelope estimates against the true envelope obtained from the analytic signal. The higher the value of  $L$ , closer is the approximation. However, there must be sufficient number of zero Fourier coefficients in the low-frequency region. The method tends to match the peaks of the envelope much more precisely than the valleys. This behavior is well known in spectral envelope modeling [9]. Although, in this example exactly one period of the signal was used for modeling, this is not necessary.

#### 5. CONCLUSION

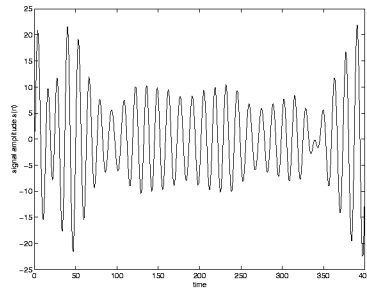
Inspired by spectral envelope modeling, we have developed an algorithm to estimate the envelope of a real-valued signal without computing the signal's Hilbert transform. It is necessary that the signal have negligible energy in the low-frequency region. We are currently developing an algorithm to compute the envelope and positive-instantaneous frequency of real-valued signals without resorting to any complex-valued computations.

#### 6. REFERENCES

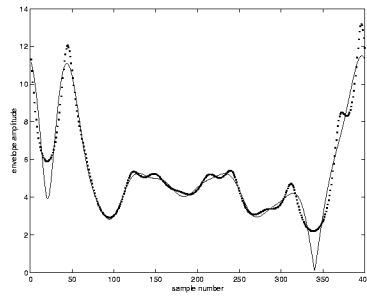
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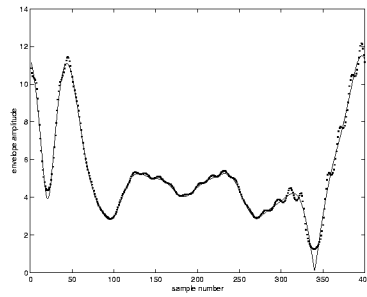
(a)



(b)



(c)



(d)

Figure 2: Fig.2(a) shows the magnitude of the Fourier coefficients used to generate the real signal  $s(t)$  shown in Fig.2(b). In Figures 2(c) and 2(d) the true envelope of the analytic signal (shown in solid line) is compared with the estimates given by  $1/|h(t)|$  shown in dotted lines. The envelope estimates shown in Fig.2(c) and Fig.2(d) correspond to  $L = 12$  and  $25$ , respectively.