MODULATING WAVEFORMS FOR OFDM

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ABSTRACT

Orthogonal frequency division multiplexing (OFDM) is a popular transmission technique that is employed in applications such as Digital Audio Broadcasting, Asymmetric Digital Subscriber Line and wireless LAN. In this work we consider design of modulating waveforms for OFDM in the presence of the delay spread and system impairments such as frequency offset and timing mismatch. We give a complete parameterization of OFDM modulating waveforms. Increasing robustness of OFDM to frequency offsets requires using long modulating waveforms. To make the implementation of OFDM systems with long modulating waveforms feasible we propose fast implementation algorithms. Some preliminary modulating waveform design examples are presented. The presented waveforms demonstrate that the robustness of OFDM systems to impairments can be improved by allowing certain degradation of unnecessarily good performances of the state of the art OFDM systems in ideal operating conditions.

1. PRINCIPLES OF OFDM

Consider a communication system which transmits a symbol stream a, with a time interval τ_0 using a modulating waveform φ. Transmitted signal, s, is a linear combination of translates of φ, $s(t) = \sum_m a[m]\varphi(t - m\tau_0)$. In the presence of multipath propagation the received signal, s_r, is a superposition of delayed and attenuated replicas of the transmitted signal, $s_r(t) = \sum_i r_i s(t - d_i)$. Note that the propagation parameters r_i and d_i in general vary with time. If the intersymbol interval τ_0 is significantly shorter than the possible delay spread, $d = \max_{i,j} |d_i - d_j|$, this multipath propagation causes severe intersymbol interference. An immediate way to cope with this problem would be to transmit symbols with time intervals larger than the delay spread, while using some form of parallelism, e.g. frequency multiplexing, to maintain the symbol rate in a given range. This requires dividing the symbol sequence into a number, N, of subsequences and as many multiplexing waveforms. With this multiplexing the transmitted signal takes the form

$$s(t) = \sum_{m=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k[m]\varphi_k(t - mK\tau_0), \qquad (1)$$

where a_k and φ_k, $k = 0, 1, \dots N - 1$, denote the symbol subsequences and the modulating waveforms respectively. In order to make this transmission scheme effective, the translation parameter K, we will call it the *interframe interval*, has to be larger than the number of multiplexed subsequences, $K > N$. If this new interval, $K\tau_0$, exceeds the length of the modulating waveforms by more than the delay spread and if in addition modulating waveforms occupy nonoverlapping frequency bands, then it is possible to decompose the received signal into components $s_{k,m}(t) = a_k[m]\sum_i r_i\varphi_k(t-d_i-mK\tau_0)$, and thus eliminate intersymbol interference using only linear band-pass filtering. This transmission scheme eliminates intersymbol interference based on good time-frequency separation between modulating waveforms and their translates. This separation, however, can only be attained at low transmission efficiencies, characterized by the ratio N/K. The idea of orthogonal frequency division multiplexing [1, 2] is to multiplex symbol sequences based on mutual orthogonality between modulating waveforms, and thus improve bandwidth efficiency by allowing spectral overlap between the channels.

In this paper we study modulating waveforms for orthogonal frequency division multiplexing. The discrete-time system model used in the sequel is described by the following two formulae, which give the multiplexed transmitted signal and the received signal as

$$s[n] = \sum_{m=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k[m]\varphi_k[n - mK], \qquad (2)$$

and

$$s_r[n] = \sum_{i=0}^{R} r_i s[n - d_i]. \qquad (3)$$

For notation convenience we assume that the sampling interval and τ_0 are equal to 1, and that $r_0 = 1$ and $d_0 = 0$.

Modulating waveforms for OFDM need to satisfy two main requirements:

1) translates of modulating waveforms by integral multiples of the interframe interval need to be mutually orthogonal,

$$\sum_n \varphi_i[n - lK]\varphi_j[n - mK] = \delta[i - j]\delta[l - m]; \qquad (4)$$

2) fast implementation should be feasible.

The orthogonality condition in (4) suggests detection of transmitted symbols as

$$\tilde{a}_l[m] = \sum_n s_r[n]\varphi_l[n - mK]. \qquad (5)$$

Due to the multipath propagation, the detected symbol $\tilde{a}_l[m]$ is different from the corresponding transmitted symbol and they are related as

$$\tilde{a}_l[m] = \phi_l[m]a_l[m] + \xi_l[m], \qquad (6)$$

where $\phi_l[m]$ is the fading factor which in principle can be compensated for using one-tap equalization, and $\xi_l[m]$ represents intersymbol interference. The intersymbol interference component

$\xi_l[m]$ is given by

$$\xi_l[m] = \sum_{i=1}^{R} r_i \sum_{(j,k)\neq(l,m)} a_j[k]\mathcal{I}(j,k,l,m,d_i), \qquad (7)$$

where the interference factors $\mathcal{I}(j,k,l,m,d_i)$ represent inner products

$$\mathcal{I}(j,k,l,m,d_i) = \sum_n \varphi_l[n-mK]\varphi_j[n-kK-d_i].$$

The goal of modulating waveform design is therefore to minimize these interference factors under the two main design requirements.

Modulating waveforms in the form of complex exponentials

$$\varphi_k[n] = (1/\sqrt{N})\exp(j2\pi kn/N),\ 0 \leq n \leq N-1,$$

that satisfy the orthogonality condition in (4) and provide the convenience of Fast Fourier Transform based implementation are an immediate solution. However, due to their bad frequency localization, with these modulating waveforms multipath propagation causes severe interference between symbols across different OFDM channels. An incredibly simple but very effective way to deal with this problem, proposed in [3], is to use for detection another set of waveforms, as

$$\tilde{a}_l[m] = \sum_n s_r[n]\psi_l[n-mK], \qquad (8)$$

where ψ_k are K samples long complex exponentials

$$\psi_k[n] = (1/\sqrt{N})\exp(j2\pi kn/N),\ 0 \leq n \leq K-1,$$

Since ψ_k is orthogonal to translates of all modulating waveforms $\varphi_l,\ l \neq k$, of the form $\varphi_l[n-mK-d_i]$ for all delay parameters in the range $0 \leq d_i \leq K-N$, in this manner intersymbol interference is completely eliminated if the delay spread does not exceed the guard interval $T_g = K-N$. This detection algorithm is referred to as the *cyclic prefix*. The problem with this approach to modulation and detection is that the system performance is drastically degraded in the presence of timing mismatch or frequency offsets, or if the delay spread exceeds the guard interval. In this paper we investigate modulating waveforms in the form of windowed complex exponentials with no restrictions on window length other than those imposed by overall processing delay. The motivation is to search through the complete set of OFDM windows for those which allow certain intersymbol interference in ideal operating conditions in exchange for improved performances in the presence of system impairments.

2. PARAMETERIZATION OF OFDM WINDOWS

The requirement for orthogonality between modulating waveforms given in (4) in the case when the waveforms are modulated versions of a given window

$$\varphi_k[n] = v[n]\exp(j2\pi kn/N) \qquad (9)$$

is equivalent to the following set of conditions on the window function:

$$\sum_i v[n+iN]v[n+iN+jK] = \frac{1}{N}\delta[j],\ n = 0,1,\ldots N-1. \qquad (10)$$

The complete set of solutions to this system of equations was first given in [4], in a closed form, in the context of short-time Fourier analysis. Actually it turns out that the same set of constraints describes windows which give the so called tight Weyl-Heisenberg frames, that are used as the tool of short-time Fourier analysis. It is interesting to note that while the modulating waveforms and their translates used for OFDM are orthogonal and span proper subspaces of $\ell^2(\mathbf{Z})$, corresponding Weyl-Heisenberg frames are linearly dependent, redundant families of vectors in $\ell^2(\mathbf{Z})$.

The complete parameterization of windows that satisfy the system of constraints in (10) will be given based on the OFDM multiplexer polyphase representation. For that purpose it is convenient to represent the multiplexer output signal s given by (2) in terms of its K polyphase components as

$$S(z) = \sum_{l=0}^{K-1} S_l(z^K)z^{-l},\ \text{where}\ S_l(z) = \sum_{n=-\infty}^{\infty} s[nK+l]z^{-n}.$$

The polyphase components of s are related to input data sequences as

$$[S_0(z)\ldots S_{K-1}(z)]^T = \mathbf{M}(z)[A_0(z)\ldots A_{N-1}(z)]^T. \qquad (11)$$

where $\mathbf{M}(z)$ is a $K \times N$ polynomial matrix that is the polyphase representation of the multiplexer and $A_i(z) = \sum_n a_i[n]z^{-n}$. The matrix $\mathbf{M}(z)$ has the form $\mathbf{M}(z) = \mathbf{V}(z)\mathbf{F}_N$, where $\mathbf{F}_N$ is the N-point discrete-Fourier transform matrix, and $\mathbf{V}(z)$ is the $K \times N$ matrix given as follows. Let M be the least common multiple of K and N, and J and L the two integers satisfying $JK = LN = M$. Consider the M-component polyphase representation of the window v,

$$V(z) = \sum_{j=0}^{M-1} z^{-j}V_j(z^M),\quad V_j(z) = \sum_{n=-\infty}^{\infty} v[nM+j]z^{-n}.$$

The row l, $0 \leq l \leq K-1$, of $\mathbf{V}(z)$ has J nonzero entries and these are polynomials $z^{-p}V_{pK+l}(z^J)$, $p = 0,1,\ldots J-1$. The polynomial $z^{-p}V_{pK+l}(z^J)$ is in the column $k(l,p)$ where $k(l,p)$ is the number satisfying $qN + k(l,p) = pK + l$, for some integer q, $0 \leq q \leq L-1$.

Example 1 *K and N are coprime, ($K = 3$ and $N = 2$).*

$$\mathbf{V}(z) = \begin{bmatrix} V_0(z^2) & z^{-1}V_3(z^2) \\ z^{-1}V_4(z^2) & V_1(z^2) \\ V_2(z^2) & z^{-1}V_5(z^2) \end{bmatrix}.$$

Example 2 *K and N have a common factor other than N, ($K = 6$ and $N = 4$). The matrix $\mathbf{V}(z)$ in this case has the form*

$$\begin{bmatrix} V_0(z^2) & 0 & z^{-1}V_6(z^2) & 0 \\ 0 & V_1(z^2) & 0 & z^{-1}V_7(z^2) \\ z^{-1}V_8(z^2) & 0 & V_2(z^2) & 0 \\ 0 & z^{-1}V_9(z^2) & 0 & V_3(z^2) \\ V_4(z^2) & 0 & z^{-1}V_{10}(z^2) & 0 \\ 0 & V_5(z^2) & 0 & z^{-1}V_{11}(z^2) \end{bmatrix}.$$

It can be shown that the modulating waveforms in (9) satisfy the orthogonality condition given by (4) if and only if the corresponding matrix $\mathbf{V}(z)$ is paraunitary, $\mathbf{V}^T(z^{-1})\mathbf{V}(z) = (1/N)\mathbf{I}_N$, ($\mathbf{I}_N$ denotes the $N \times N$ identity matrix). If N and K are not coprime $\mathbf{V}(z)$ is not full, so it is paraunitary if and only if a particular set of N/J of its submatrices are paraunitary. As an illustration, consider again the two examples given above.

Example 3 *The matrix* $\mathbf{V}(z)$ *in Example 1 is paraunitary if and only if the matrix*

$$\mathbf{U}(z) = \begin{bmatrix} V_0(z) & V_3(z) \\ V_4(z) & zV_1(z) \\ V_2(z) & V_5(z) \end{bmatrix} \quad (12)$$

is paraunitary.

Example 4 *The matrix* $\mathbf{V}(z)$ *in Example 2 is paraunitary if and only if the matrices*

$$\mathbf{U}_0(z) = \begin{bmatrix} V_0(z) & V_6(z) \\ V_8(z) & zV_2(z) \\ V_4(z) & V_{10}(z) \end{bmatrix} \quad \mathbf{U}_1(z) = \begin{bmatrix} V_1(z) & V_7(z) \\ V_9(z) & zV_3(z) \\ V_5(z) & V_{11}(z) \end{bmatrix}$$

are paraunitary.

In general, the $M = \text{LCM}(K, N)$ polyphase components of a window for an N channel OFDM with K point interframe interval are given up to time delays as entries of N/J paraunitary matrices of size $L \times J$, and vice versa. Parameterizations of paraunitary matrices have been previously studied in the filter bank literature [5].

Example 5 *In the case of of* $N = 128$ *channel OFDM with* $K = 192$ *point interframe interval the* $M = 384$ *polyphase components of the corresponding window are given as entries of* 64 *paraunitary* 3×2 *matrices as*

$$\mathbf{U}_i(z) = \begin{bmatrix} V_{0\cdot 64+i}(z) & V_{3\cdot 64+i}(z) \\ V_{4\cdot 64+i}(z) & zV_{1\cdot 64+i}(z) \\ V_{2\cdot 64+i}(z) & V_{5\cdot 64+i}(z) \end{bmatrix}, \quad i = 0, 1, \ldots 63.$$

Observe the delay factors next to the entries $V_{1\cdot 64+i}$ *in the above matrices, which result in two* 64*-tap long zero segments in the corresponding window.*

2.1. Fast Implementation

Fast implementation algorithms for OFDM follow directly from the polyphase representation of the multiplexer. Observe that the polyphase components of the multiplexed signal are given as

$$[S_0(z) \ldots S_{K-1}(z)]^T = \mathbf{V}(z)[C_0(z) \ldots C_{N-1}(z)]^T \quad (13)$$

where $C_k(z) = \sum_n c_k[n]z^{-n}$, and the sequences c_k, are obtained from the input symbol sequences by applying discrete Fourier transform,

$$[c_0[n]\; c_1[n]\; \ldots\; c_{N-1}[n]]^T = \mathbf{F}_N[a_0[n]\; a_1[n]\; \ldots\; a_{N-1}[n]]^T. \quad (14)$$

Each row of $\mathbf{V}(z)$ contains only J nonzero elements that are $L/M-1$ order polynomials, where L is the window length. Computing a point in the sequence s from sequences c_i thus requires L/K multiplications and $L/K - 1$ additions. The overall numerical complexity of this multiplexer implementation per point of the output sequence is

$$\frac{L}{K}\text{multiplications} + \left(\frac{L}{K} - 1\right) \text{additions} + \frac{\mathcal{C}(FFT_N)}{K}, \quad (15)$$

where $\mathcal{C}(FFT_N)$ denotes complexity of an N-point FFT algorithm. Note that we consider real valued windows, and that sequences c_i are complex, so multiplication here means multiplication of a complex number by a real number, and addition refers to complex addition. Complexity of an N point FFT algorithm is $O(N log_2 N)$ complex multiplications and additions.

One of the reasons for which long windows, i.e. longer than K taps, were not used for OFDM was that polyphase multiplexer representations were not known for cases with rational K/N ratios, and hence it was not clear that fast implementations based on FFT were possible.

3. WAVEFORM DESIGN EXAMPLES

In this section we present several design examples and compare them based on a measure of intersymbol interference. As the measure of intersymbol interference we consider the sum of squares of all individual components in the expression for intersymbol interference in (7), assuming that all transmitted symbols are equal to 1 and assuming a two-ray multipath propagation model

$$s_r[n] = s[n] + s[n-d]. \quad (16)$$

This gives some average measure of interference for the given multipath propagation model, independent of the chosen pair of indices l and m and it has the form

$$\mathcal{I}_a = \sum_{(j,k)\neq(l,m)} |\mathcal{I}(j,k,l,m,d)|^2. \quad (17)$$

A window, v_{160}, for $N = 128$ channel OFDM with $K = 160$ point interframe interval is shown in Figure 1. The total length of v_{160} is $L = 1024$, however only 640 of its taps are different from zero, and that number is relevant for the implementation complexity. Figure 2 shows intersymbol interference plots calculated for

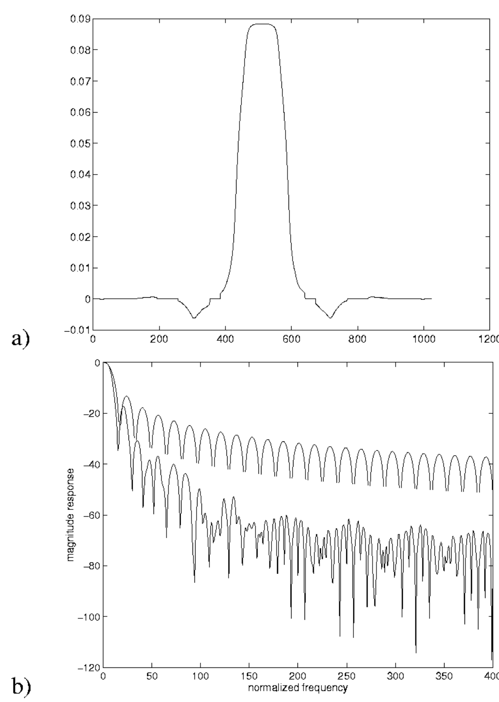

Figure 1: Window v_{160} for 128 channel OFDM with 160 point interframe interval. a) Time domain plot. b) Lin-log plots of magnitude responses of v_{160} and the 128-tap rectangular window.

v_{160} (solid lines) together with intersymbol interference plots for the length $L = 128$ rectangular window with cyclic prefix detection (dashed lines), for the two-ray propagation model. For each of the windows there are two plots in the figure, one of which gives intersymbol interference assuming no frequency offset, and the other assuming 5% frequency offset. The plots that correspond to the case with no frequency offset (those with lower intersymbol interference) demonstrate the absence of intersymbol interference for the cyclic prefix detection when the delay spread is smaller than a certain value, as well as the dramatic degradation of its performance as soon as the delay spread exceeds that limit, which is in this case equal to $(K - N)/2$. Note that the cyclic prefix detection can be set to work perfectly for the delay spreads up to the length of the guard interval $T_g = K - N$, but in that case even one point timing mismatch increases intersymbol interference to $-18dB$. All plots shown in this paper represent the cyclic prefix detection set to be maximally robust to timing mismatch, which in turn reduces the range of the delay spread for which this technique exhibits excellent performances. Even with no frequency offset one can observe $4 - 6dB$ improvement attained with v_{160} for a wide range of delay spread values beyond $T_g/2$. The plots obtained for 5% frequency offset demonstrate clear improvement attained with v_{160}, which is $0 - 4dB$ for delay spreads smaller than $T_g/2$, and $4 - 6dB$ for larger delay spreads.

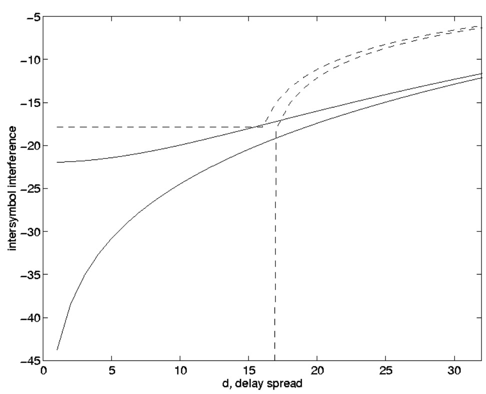

Figure 2: Intersymbol interference, $20 \log_{10} \mathcal{I}_a$, for OFDM with v_{160} (solid lines) and for the cyclic prefix detection (dashed lines). The plots show intersymbol interference for the two-ray propagation model in (16), calculated for the cases with no frequency offset (curves with lower intersymbol interference) and with 5% frequency offset.

A window, v_{192}, for $N = 128$ channel OFDM with $K = 192$ point interframe interval is shown in Figure 3. Intersymbol interference plots for OFDM with this window and the cyclic prefix detection with $T_g = 64$ point guard interval are shown in Figure 4. The improvement of robustness over the cyclic prefix detection is $5 - 7dB$ for delay spreads larger than T_g when there is no frequency offset. In the presence of 5% frequency offset the improvement is $0 - 9dB$ for $d < T_g/2$ and $5 - 7dB$ for $d > T_g/2$. Observe also from Figure 2 and Figure 4 that increasing the interframe interval from $K = 160$ to $K = 192$ provides around $5dB$ reduction of intersymbol interference, when OFDM is based on windows v_{160} and v_{192}. This is of course paid by the corresponding loss in transmission efficiency.

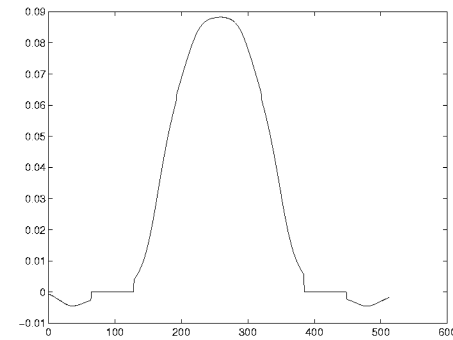

Figure 3: Window v_{192} for 128 channel OFDM with 192 point interframe interval.

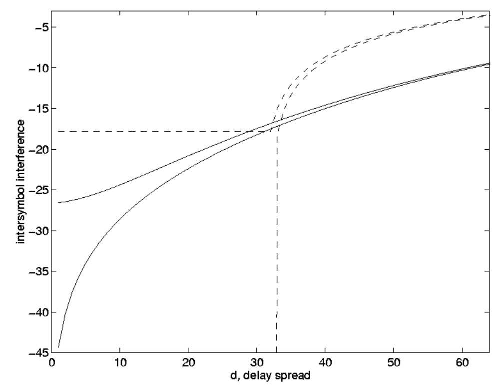

Figure 4: Intersymbol interference, $20 \log_{10} \mathcal{I}_a$, for 128 channel OFDM with 192 point interframe interval, for window v_{192} (solid lines) and for the cyclic prefix detection (dashed lines). The plots show intersymbol interference for the two-ray propagation model in (16), for the cases with no frequency offset (curves with lower intersymbol interference) and with 5% frequency offset.

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