PERFORMANCE MEASURES FOR ESTIMATING VECTOR SYSTEMS

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ABSTRACT

We propose a framework of performance measures for analyzing estimators of geometrical vectors that have intuitive physical interpretations, are independent of the coordinate frame and parameterization, and have no artificial singularities. We obtain finite-sample and asymptotic lower bounds on them for large classes of estimators and show how they may be used as system design criteria. We determine a simple asymptotic relationship that is applicable to both the measures and their bounds.

1. INTRODUCTION

The need to estimate a (3D) geometrical vector quantity, i.e. one whose magnitude and direction are meaningful quantities in the context of the application, arises in many signal processing problems. For example, determination of a vector representing location or bearing is the focus of many radar [1], sonar [2], and mobile communications problems [3], a velocity vector is required in target tracking, and a vector dipole moment is an unknown of interest in magnetoencephalography [4]. For an estimate of such a vector, the usual mean-square error (MSE) matrix, which indicates the variability in the individual coordinates, may not be a very intuitive measure of performance. Furthermore, it is dependent upon the choice of reference coordinate frame. More generally, the unknown vector is described by a parameter vector $\boldsymbol{\theta}$ that itself has no physical interpretation and may contain artificial singularities, e.g. spherical coordinates. Again, the MSE matrix of the parameter vector is dependent on the reference coordinate frame, and may become non-singular because of the singularities.

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In this paper, we construct a unified framework for the analysis of errors that occur in estimating a vector through a set of geometrically-based error measures. These measures are more physically appealing and intuitive than the MSE matrix, do not contain artificial singularities, and are referenced to the unknown vector, so ensuring they are independent of rotations in the observer's coordinate system. We consider three error measures: mean-square error length (MSEL), mean-square angular error (MSAE), and mean-square range error (MSRE). We derive lower bounds on the asymptotic normalized version of these quantities, holding for large classes of estimators, that are expressed in terms of the Cramér-Rao bound (CRB) on the parameter vector $\boldsymbol{\theta}$. The classes of estimators for which they hold are described in terms of conditions on the bias and bias gradient of each estimator. We also show that these bounds have a finite sample equivalent for the class of unbiased estimators. We obtain simplified expressions for orthogonal curvilinear parameterizations of the geometrical vector and illustrate the results using spherical coordinates. The analysis is performed under the assumptions of both known (e.g. finding the direction to a far-field source) and unknown (e.g. finding the direction and range of a near-field source) length. A simple relationship is derived between the error measures that holds asymptotically. Finally, we discuss the use of these measures and their bounds for system design.

The MSAE was introduced in [1] for a unit length vector expressed in spherical coordinates and an asymptotic bound was derived. The MSEL for unbiased estimators was considered in [4]. Note that although geometrical interpretations of vectors are most obvious in three dimensions, the error measures we examine throughout this paper are still useful for their coordinate independence and lack of artificial singularities. Indeed our results are applicable in an arbitrary number of dimensions.

2. MODELS, MEASURES, AND BOUNDS

We assume that we have N measurement snapshots $\{\boldsymbol{y}_i\}_{i=1}^N$ that are independent identically distributed (i.i.d) with a distribution that is parameterized by a vector $\boldsymbol{\eta}$. Let us suppose that the vector quantity of interest $\boldsymbol{v} \in \mathbb{R}^d$ is a function of a subset $\boldsymbol{\theta}$ of these parameters. Therefore, as far as we are concerned the remaining parameters are nuisance parameters. For example, if we are interested in the location of a moving radar target, the parameters of interest are the azimuth and elevation of the return and its time delay. The Doppler shift, attenuation coefficient, and any parameters that describe the noise statistics are nuisance parameters [5]. We suppose that the distribution of $\boldsymbol{y}$ and its parameterization satisfies enough regularity conditions so that the Fisher Information Matrix (FIM) exists and is non-singular.

2.1. Mean-Square Error Length

Suppose $\hat{\boldsymbol{v}} \in \mathbb{R}^d$ is an estimator of $\boldsymbol{v}$. The mean-square error length (MSEL) is defined as the mean-square value of the length of the error vector,

$$\text{MSEL} \triangleq \text{E}\,|\Delta\boldsymbol{v}|^2, \tag{2.1}$$

where $\Delta\boldsymbol{v} = \triangleq \hat{\boldsymbol{v}} - \boldsymbol{v}$. It provides a single, overall measure of estimation performance, has no artificial singularities, and is independent of the reference coordinate frame.

Let $\hat{\boldsymbol{v}}$ have expected value $\text{E}\,\hat{\boldsymbol{v}} \triangleq \bar{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon}$ represents bias, and we have suppressed dependence on $\boldsymbol{\theta}$ for simplicity of notation. The model and estimator must be such that $\boldsymbol{v}$ and $\boldsymbol{\epsilon}$ are differentiable with respect to $\boldsymbol{\theta}$ and the mild regularity conditions [7], pp. 65, are satisfied. Then the covariance matrix of $\hat{\boldsymbol{v}}$ satisfies

$$\text{E}(\hat{\boldsymbol{v}} - \bar{\boldsymbol{v}})(\hat{\boldsymbol{v}} - \bar{\boldsymbol{v}})^T \geq \frac{\partial\bar{\boldsymbol{v}}}{\partial\boldsymbol{\theta}}\text{CRB}(\boldsymbol{\theta})\frac{\partial\bar{\boldsymbol{v}}}{\partial\boldsymbol{\theta}}^T, \tag{2.2}$$

where $\text{CRB}(\boldsymbol{\theta})$ is the block of $\text{CRB}(\boldsymbol{\eta})$ corresponding to those parameters that describe $\boldsymbol{v}$. Of course, the entries of $\text{CRB}(\boldsymbol{\theta})$ are generally dependent on the nuisance parameters. Note that [8] gives a procedure to calculate $\text{CRB}(\boldsymbol{\theta})$ that avoids inversion of the full FIM and instead only requires inversion of a matrix equal to the dimension of $\boldsymbol{\theta}$. The mean-square error matrix thus satisfies

$$\text{E}\,\Delta\boldsymbol{v}\Delta\boldsymbol{v}^T \geq \frac{\partial\bar{\boldsymbol{v}}}{\partial\boldsymbol{\theta}}\text{CRB}(\boldsymbol{\theta})\frac{\partial\bar{\boldsymbol{v}}}{\partial\boldsymbol{\theta}}^T + \boldsymbol{\epsilon}\boldsymbol{\epsilon}^T. \tag{2.3}$$

The MSEL is just the trace of $\text{E}\,\Delta\boldsymbol{v}\Delta\boldsymbol{v}^T$ so

$$\text{MSEL} \geq \text{tr}\left\{\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{\theta}}\text{CRB}(\boldsymbol{\theta})\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{\theta}}^T\right\} + |\boldsymbol{\epsilon}|^2 + \text{O}\left(\frac{\partial\boldsymbol{\epsilon}}{\partial\boldsymbol{\theta}}\text{CRB}(\boldsymbol{\theta})\right). \tag{2.4}$$

We now suppose the bias satisfies

$$\begin{aligned}\boldsymbol{\epsilon} &= \text{o}(1/\sqrt{N}) \\ \partial\boldsymbol{\epsilon}/\partial\boldsymbol{\theta} &= \text{o}(1).\end{aligned} \tag{2.5}$$

Since the data consists of i.i.d. snapshots, $\text{CRB}(\boldsymbol{\theta})$ is proportional to $1/N$. Therefore, under condition (2.5)

$$\text{MSEL} \geq \text{tr}\left\{\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{\theta}}\text{CRB}(\boldsymbol{\theta})\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{\theta}}^T\right\} + \text{o}(1/N). \tag{2.6}$$

Condition (2.5) essentially describes the class of all asymptotically unbiased estimators whose asymptotic performance is limited by their stochastic variability rather than their bias and whose expected values exhibit a certain degree of smoothness with respect to changes in the unknown parameter.

Define the asymptotic normalized mean-square error length as

$$\text{MSEL}_\infty \triangleq \lim_{N\to\infty} N\{\text{E}\,|\Delta\boldsymbol{v}|^2\}. \tag{2.7}$$

Normalizing and taking limits of (2.6) as $N \to \infty$ we see that for any estimator satisfying (2.5)

$$\text{MSEL}_\infty \geq \text{MSEL}_\text{B} \triangleq N\,\text{tr}\left\{\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{\theta}}\text{CRB}(\boldsymbol{\theta})\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{\theta}}^T\right\}. \tag{2.8}$$

Note that this bound is independent of N, assuming i.i.d. snapshots. Since it is derived from the CRB, it will be tight for any asymptotically efficient estimator. There is an analogous finite-sample result for the smaller class of unbiased estimators. Specifically, if $\hat{\boldsymbol{v}}$ is a (locally) unbiased estimator of $\boldsymbol{v}$, i.e $\boldsymbol{\epsilon}$ and $\partial\boldsymbol{\epsilon}/\boldsymbol{\theta}$ are zero for all values (in a neighborhood) of $\boldsymbol{\theta}$, then $\text{MSEL} \geq \text{MSEL}_\text{B}/N$. Note that the asymptotic bound is equal to the finite-sample bound for a single snapshot.

2.1.1. Orthogonal Curvilinear Parameterizations

Most common parameterizations of a 3D vector, including spherical coordinates, circular and elliptic cylinder coordinates, parabolic coordinates, and prolate spheroidal coordinates, are examples of orthogonal curvilinear coordinate systems. Such a parameterization is characterized by the fact that its Jacobian $\partial \boldsymbol{v}/\partial \boldsymbol{\theta}$ has orthogonal columns. Therefore, if we denote the j,kth entry of $\mathrm{CRB}(\boldsymbol{\theta})$ by C_{jk}, equation (2.8) simplifies to

$$\mathrm{MSEL_B} = N \sum_{j=1}^{3} h_j^2 C_{jj} \qquad (2.9)$$

where $h_j = |\partial \boldsymbol{v}/\partial \theta_j|$ is called the scale factor for the ith coordinate [9]. For example, in spherical coordinates (2.9) becomes

$$\mathrm{MSEL_B} = N\{\mathrm{CRB}(r) + r^2 \cos^2 \psi \mathrm{CRB}(\phi) + r^2 \mathrm{CRB}(\psi)\}, \qquad (2.10)$$

where r, ϕ, and ψ are the length, azimuth, and elevation, respectively.

2.2. Mean-Square Angular Error

In some applications, e.g. radar or sonar target localization, separate characterizations of the error in the bearing and the range can be very informative. In this section we consider the mean-square angular error (MSAE), introduced in [1] for a unit vector in spherical coordinates and used as an array design criterion in [10]. The MSAE is defined as $\mathrm{E}\,\gamma^2$, where $\gamma \in [0, \pi]$ is the angle between $\hat{\boldsymbol{v}}$ and $\boldsymbol{v}$.

Consider the normalized vector $\boldsymbol{u} \triangleq \boldsymbol{v}/r$ and the normalized estimate $\hat{\boldsymbol{u}} \triangleq \hat{\boldsymbol{v}}/\hat{r}$, where r and $\hat{r}$ are the lengths of $\boldsymbol{v}$ and $\hat{\boldsymbol{v}}$, respectively. Now

$$\gamma = 2 \sin^{-1} \frac{|\Delta \boldsymbol{v}|}{2}, \qquad (2.11)$$

where $\Delta \boldsymbol{u} = \hat{\boldsymbol{u}} - \boldsymbol{u}$. Using a standard expansion for $\sin^{-1}$ the squared angular error is

$$\gamma^2 = |\Delta \boldsymbol{u}|^2/r^2 + \mathrm{O}(|\Delta \boldsymbol{u}|^4). \qquad (2.12)$$

All terms in the expansion are positive so truncating the series at any point gives a lower bound on γ^2. In particular $|\Delta \boldsymbol{u}|^2$ is a lower bound for γ^2. Geometrically speaking, $\Delta \boldsymbol{u}$ is the chord of the unit circle joining the tips of $\boldsymbol{u}$ and $\hat{\boldsymbol{u}}$, and $r^2\gamma^2$ is the arc length. Therefore, assuming that $\mathrm{E}\,|\Delta \boldsymbol{u}|^4$, and all higher order moments of $|\Delta \boldsymbol{u}|$, are $\mathrm{o}(1/N)$,

$$\mathrm{MSAE} = \mathrm{E}\,|\Delta \boldsymbol{u}|^2 + \mathrm{o}(1/N). \qquad (2.13)$$

The moment condition is clearly satisfied when $\mathrm{E}\,|\Delta \boldsymbol{v}|^4$ is $\mathrm{o}(1/N)$, however, this is not necessary; consider, for example, an estimator whose error is asymptotically due to inaccuracies in range rather than bearing. Following similar reasoning to Section 2.1,

$$\mathrm{E}\,|\Delta \boldsymbol{u}|^2 \geq \mathrm{tr}\left\{ \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\theta}} \mathrm{CRB}(\boldsymbol{\theta}) \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\theta}}^T \right\} + \mathrm{o}(1/N), \qquad (2.14)$$

if $\boldsymbol{\epsilon}_{\mathrm{u}} = \mathrm{o}(1/\sqrt{N})$ and $\partial \boldsymbol{\epsilon}_{\mathrm{u}}/\partial \boldsymbol{\theta} = \mathrm{o}(1)$, where $\boldsymbol{\epsilon}_{\mathrm{u}}$ is the bias of $\hat{\boldsymbol{u}}$ considered as an estimate of $\boldsymbol{u}$. Condition (2.5) is sufficient, but again not necessary, for this to hold. Therefore, defining the asymptotic normalized MSAE and its bounds in the obvious way, we have that

$$\begin{aligned} \mathrm{MSAE}_\infty &\triangleq \lim_{N\to\infty} N\,\mathrm{E}\,\gamma^2 \geq \mathrm{MSAE_B} \\ &\triangleq N\,\mathrm{tr}\left\{ \frac{\partial(\boldsymbol{v}/r)}{\partial \boldsymbol{\theta}} \mathrm{CRB}(\boldsymbol{\theta}) \frac{\partial(\boldsymbol{v}^T/r)}{\partial \boldsymbol{\theta}} \right\}. \end{aligned} \qquad (2.15)$$

This bounds applies to, at least, all estimators satisfying (2.5) and having a fourth order moment $\mathrm{E}\,|\Delta \boldsymbol{v}|^4 = \mathrm{o}(1/N)$. The finite-snapshot version of (2.15) requires that $\mathrm{E}(\hat{\boldsymbol{v}}/\hat{r})$ be an unbiased estimator of $\boldsymbol{v}/r$, in which case the MSAE is bounded by $\mathrm{MSAE_B}/N$ for all N. This is not implied by the unbiasedness of $\hat{\boldsymbol{v}}$, so, in general the bounds on the MSEL and on the MSAE do not simultaneously hold for a finite number of snapshots.

Using spherical coordinates, for example, (2.15) becomes

$$\mathrm{MSAE_B} = N\{\cos^2 \psi \mathrm{CRB}(\phi) + \mathrm{CRB}(\psi)\}. \qquad (2.16)$$

This is the expression obtained in [1] for a unit length estimate of a unit length vector. However, note that although r does not appear in (2.16) it is not necessarily independent of the length of $\boldsymbol{v}$, since $\mathrm{CRB}(\phi)$ and $\mathrm{CRB}(\psi)$ may depend on it.

If the length of $\boldsymbol{v}$ is known, e.g. the distance to the target is known or the source is far-field so that $\boldsymbol{v}$ is taken to be a unit vector, then it is natural to constrain $\hat{\boldsymbol{v}}$ to have the same length. In that case, $\Delta \boldsymbol{u} = \Delta \boldsymbol{v}/r$ and it follows that, under (2.5),

$$\mathrm{MSAE}_\infty = \mathrm{MSEL}_\infty/r^2, \qquad (2.17)$$

if $\mathrm{E}\,|\Delta \boldsymbol{v}|^4 = \mathrm{o}(1/N)$, with a similar expression for the bound $\mathrm{MSAE_B}$ in terms of $\mathrm{MSEL_B}$.

2.3. Mean-Square Range Error

The MSRE is defined as

$$\mathrm{MSRE} \triangleq \mathrm{E}\,\Delta r^2, \qquad (2.18)$$

where $\Delta r = \hat{r} - r$. By considering $\hat{r}$ as an estimate of r with bias ϵ_r, we can follow a similar development to Section 2.1 to find that

$$\mathrm{MSRE} \geq \frac{\partial r}{\partial \boldsymbol{\theta}^T} \mathrm{CRB}(\boldsymbol{\theta}) \frac{\partial r}{\partial \boldsymbol{\theta}} + \mathrm{o}(1/N), \qquad (2.19)$$

assuming that ϵ_r is $\mathrm{o}(1/\sqrt{N})$ and $\partial \epsilon_r/\partial \boldsymbol{\theta}$ is $\mathrm{o}(1)$. It can be shown that these are satisfied if (2.5) holds and, in addition,

$\mathrm{E}(|\Delta \boldsymbol{v}|^2) = \mathrm{o}(1/\sqrt{N})$ and $\partial\, \mathrm{E}\, |\Delta \boldsymbol{v}|^k / \partial \boldsymbol{\theta} = \mathrm{o}(1)$ for all $k \geq 2$ [11]. Defining the asymptotic normalized MSRE in the obvious manner, (2.19) shows that

$$\mathrm{MSRE}_\infty \geq \mathrm{MSRE}_{\mathrm{B}} \triangleq N \frac{\partial r}{\partial \boldsymbol{\theta}^T} \mathrm{CRB}(\boldsymbol{\theta}) \frac{\partial r}{\partial \boldsymbol{\theta}}. \qquad (2.20)$$

Furthermore, if $\hat{r}$ is an unbiased estimate of r, then $\mathrm{MSRE} \geq \mathrm{MSRE}_{\mathrm{B}}/N$ for all N. For the spherical coordinate system $\mathrm{MSRE}_{\mathrm{B}} = N\mathrm{CRB}(r)$.

2.4. Asymptotic Relationships and System Design

We have already seen that a simple relationship (2.17) exists between $MSAE_\infty$ and $MSEL_\infty$ (and between their bounds $\mathrm{MSAE}_{\mathrm{B}}$ and $\mathrm{MSEL}_{\mathrm{B}}$) when r is known. When the length of $\boldsymbol{v}$ is unknown, a relationship can be obtained between $MSEL_\infty$, $MSAE_\infty$, and MSRE_∞ for any estimator such that $E(|\Delta \boldsymbol{v}|^4)$, and higher moments, are $\mathrm{o}(1/N)$. In [11] we show that this relationship is

$$\mathrm{MSEL}_\infty = \mathrm{MSRE}_\infty + r^2 \mathrm{MSAE}_\infty \qquad (2.21)$$

A similar relationship holds for the bounds on these quantities. However, $\hat{\boldsymbol{v}}$ does not need to be asymptotically efficient, in the sense that the performance measures we discuss attain their bounds as $N \to \infty$, in order for (2.21) to hold.

2.5. System Design

The bound $\mathrm{MSEL}_{\mathrm{B}}$ provides a single algorithm-independent measure of performance and as such is a suitable criterion function for system design. Since it depends on $\boldsymbol{\theta}$ (and possibly other nuisance parameters) we can either minimize it for a particular *a priori* likely value of $\boldsymbol{\theta}$ as in [4], or we can take a Bayesian approach and minimize a weighted average over the parameter space. Equation (2.21) shows that it is just a particular linear combination of $\mathrm{MSAE}_{\mathrm{B}}$ and $\mathrm{MSRE}_{\mathrm{B}}$, therefore, more generally we can use a linear combination of $\mathrm{MSAE}_{\mathrm{B}}$ and $\mathrm{MSRE}_{\mathrm{B}}$ chosen so as to reflect the relative importance of angle and range estimates in the application.

3. CONCLUSION

We have constructed a unified framework for the analysis of errors obtained in estimating geometrical vectors. We presented a number of error measures that are more physically meaningful than the standard covariance matrix. In addition these measures are independent of the coordinate reference frame and do not contain artificial singularities. We derived finite sample and asymptotic bounds on these measures for large classes of estimators, and showed how they are related.

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