CARDINAL MULTIWAVELETS AND THE SAMPLING THEOREM

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ABSTRACT

This paper considers the classical Shannon sampling theorem in multiresolution spaces with scaling functions as interpolants. As discussed by Xia and Zhang, for an orthogonal scaling function to support such a sampling theorem, the scaling function must be cardinal. They also showed that the only orthogonal scaling function that is both cardinal and of compact support is the Haar function, which has only 1 vanishing moment and is not continuous. This paper addresses the same question, but in the *multi*wavelet context, where the situation is different. This paper presents the construction of orthogonal multiscaling functions that are simultaneously cardinal, of compact support, and have more than one vanishing moment. The scaling functions thereby support a Shannon-like sampling theorem. Such wavelet bases are appealing because the initialization of the discrete wavelet transform (prefiltering) is the identity operator — the projection of a function onto the scaling space is given by its samples.

1. INTRODUCTION

The sampling theorem of Shannon *et al* for band-limited signals is one of the cornerstones of signal processing and communication theory. Indeed, the representation of a function by its samples is an important question with a long history. While the Shannon sampling theorem is based on band-limited signals, it is natural to investigate other signal classes for which a sampling theorem holds. The assumption that a signal is band-limited, although eminently useful, is not always realistic. Note that (*i*) band-limited signals are of infinite duration, and (*ii*) the sinc function, used to reconstruct a band-limited function from its samples, is of infinite support and decays only as $|1/x|$. Because of the importance in analyzing and detecting transients and singularities, we are particularly interested in sampling theorems for signals of finite duration, and for which the reconstruction function is also of compact support.

To this end, note that the sinc function is one of the primary examples of an orthogonal scaling function from the theory of wavelet bases. The sinc function generates a scaling space V in the context of multiresolution analysis and serves as the interpolant in the context of the sampling theorem. The question naturally arises — are there orthogonal wavelet bases for which the scaling function both (*i*) supports a sampling theorem in the same fashion and (*ii*) is of compact support? Unfortunately, the Haar scaling function is the only orthogonal scaling function of compact support for which a Shannon-like sampling property holds, as proven in [15].

This paper takes up the same question, but in the context of multiwavelet bases (wavelet bases based on more than a single scaling function), where the situation is different. In this paper it is shown, via the construction of examples, that for orthogonal multiwavelet bases it is possible for the scaling functions to achieve simultaneously the sampling property, compact support, and more than one vanishing moment.

A variety of results regarding wavelet bases and sampling theorems have been described. Walter has given a sampling theorem describing the reconstruction of a function f in a scaling space V from its samples [14]. Walter's theorem does not require that the scaling function $\phi(t)$ be cardinal (interpolatory, see below), however, the interpolant is generally not the same function as the scaling function. Aldroubi and Unser have also considered wavelet sampling and the role of cardinal scaling functions, especially in the context of biorthogonal bases [1]. The notion of scale-limited signals and the issue of translation invariance in wavelet sampling is discussed in [6].

2. PRELIMINARIES

Through the paper, t is real, and n is integer.

From the classical Shannon sampling theorem, if $f(t)$ is band-limited to $(-\pi, \pi)$ then

$$f(t) = \sum_n f(n) \operatorname{sinc}(t - n)$$

where $\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$.

An important property of the sinc function is that it is a cardinal function. A function $\phi(t)$ is said to be a *cardinal function* if

$$\phi(n) = \delta(n) = \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n = \pm 1, \pm 2, \ldots \end{cases}$$

Cardinal functions take the value 0 on the nonzero integers, and take the value 1 at $t = 0$.

From the theory of wavelet bases, a function $\phi(t)$ is said to be an *orthogonal scaling function* if

1. $\phi(t)$ satisfies a dilation equation:

$$\phi(t) = \sqrt{2} \sum_n h(n) \phi(2t - n)$$

($h(n)$ is known as the scaling filter.)

2. $\phi(t)$ is orthogonal to its integer shifts:

$$\int \phi(t)\,\phi(t-n)\,dt = \delta(n)$$

The sinc function *is* such a function, with the additional property that it is cardinal [14]. The sinc function is a *cardinal orthogonal scaling function*, or COSF.

From the theory of wavelet bases, the *scaling space* $V_j(\phi)$ associated with a scaling function $\phi(t)$ is

$$V_j(\phi) = \operatorname*{Span}_{n} \{\phi(2^j t - n)\}.$$

The Shannon sampling theorem can then be stated in the wavelet context. Let $\phi(t)$ be the sinc function. If $f(t) \in V_0(\phi)$, then

$$f(t) = \sum_n f(n)\,\phi(t-n)$$

Does the sampling theorem hold for other $\phi(t)$? It is shown in [15], that the sampling property holds for an orthogonal scaling function $\phi(t)$, if and only if $\phi(t)$ is also cardinal. Therefore, every cardinal orthogonal scaling function yields a sampling theorem. The Shannon sampling theorem for band-limited signals is the special case obtained using the sinc function.

An important question arises — do there exist cardinal orthogonal scaling functions of compact support? The answer is yes: the Haar function is one example. However, as mentioned in the Introduction, there are no others [15]. The orthogonal scaling functions of Daubechies are not cardinal. The Haar function is the only cardinal orthogonal scaling function of compact support.

2.1. Halfband Filters

It is convenient to characterize a compactly supported COSF in terms of the scaling filter h. Recall first the definition of a halfband filter. $h(n)$ is *halfband* if $h(2n) = c \cdot \delta(n)$ for some nonzero c. Halfband filters take the value 0 on the even integers, except at $n = 0$.

Let $r(n)$ denote the autocorrelation sequence of $h(n)$. $r(n) = \sum_k h(k)h(k+n)$. For $h(n)$ to generate an orthogonal scaling function $\phi(t)$, it is necessary that $r(n)$ be halfband, $r(2n) = \delta(n)$ [3]. On the other hand, for a scaling filter $h(n)$ to generate a cardinal scaling function $\phi(t)$, it is necessary that $h(n)$ be halfband, $h(2n) = \frac{1}{\sqrt{2}}\delta(n)$ [15]. Hence, for $h(n)$ to generate a scaling function that is both cardinal and orthogonal, it is necessary that both $h(n)$ and $r(n)$ be halfband.

The Haar function is the only COSF of compact support because the only appropriate FIR halfband filters $h(n)$ whose autocorrelation function are also halfband, are filters having 2 nonzero coefficients. Other examples of COSFs are given in [10, 15]. The paper [15] describes COSFs based on IIR scaling filters $h(n)$; although they are not of compact support, their decay is exponential. Note in addition that scaling functions based on scaling filters of the form $H(z) = A(z^2) + z^{-(2d+1)}$, where $A(z)$ is allpass, are cardinal because such scaling filters are halfband (up to a shift). The scaling function in Figure 2 of [10] is therefore a COSF. Although not of compact support, it's decay is exponential.

3. MULTIWAVELET BASES AND THE SAMPLING THEOREM

Multiwavelet bases have received much attention since 1994 when it was shown by example in [4, 5] that symmetry, orthogonality,

compact support and approximation order $K > 1$ can be simultaneously achieved, which is not possible in the traditional scalar wavelet case.

In this paper, we show that using multiwavelet bases it is possible to achieve simultaneously cardinality, orthogonality, compact support, and approximation order $K > 1$. That is, there exist multiwavelet orthogonal scaling functions of compact support and approximation order $K > 1$ for which a Shannon-like sampling property holds, which is not possible in the scalar wavelet case.

Multiwavelet bases are wavelet bases based on several scaling and wavelet functions. This paper considers multiwavelet bases based on 2 scaling functions $\phi_0(t)$, $\phi_1(t)$ and 2 wavelet functions $\psi_0(t)$, $\psi_1(t)$. Accordingly, there are 2 scaling filters $h_0(n)$, $h_1(n)$ and 2 wavelet filters $h_2(n)$, $h_3(n)$.

The functions $\phi_0(t)$, $\phi_1(t)$ are *orthogonal multiscaling functions* if

1. $\phi_0(t)$, $\phi_1(t)$ satisfy a *matrix* dilation equation

$$\underline{\phi}(t) = \sqrt{2}\sum_n C(n)\,\underline{\phi}(2t-n)$$

where $\underline{\phi}(t) = (\phi_0(t), \phi_1(t))^t$, and $C(n)$ are 2 by 2 matrices.

2. $\phi_0(t)$, $\phi_1(t)$ are orthogonal to their integer shifts.

$$\int \phi_i(t)\,\phi_j(t-n)\,dt = \delta(i-j)\cdot\delta(n)$$

The notation for $C(n)$ used in this paper is $[C(n)]_{i,j} = h_i(2n+j)$. For example

$$C(0) = \begin{pmatrix} h_0(0) & h_0(1) \\ h_1(0) & h_1(1) \end{pmatrix}, \quad C(1) = \begin{pmatrix} h_0(2) & h_0(3) \\ h_1(2) & h_1(3) \end{pmatrix},$$

etc, where $h_0(n)$ and $h_1(n)$ are the two scaling filters.

The scaling space $V_j(\phi_0, \phi_1)$ is given by

$$V_j(\phi_0, \phi_1) = \operatorname*{Span}_{n}\{\phi_0(2^j t - n), \phi_1(2^j t - n)\}.$$

The functions $\phi_0(t)$, $\phi_1(t)$ will be called *cardinal* if

$$\phi_0(n/2) = \delta(n)$$
$$\phi_1(n/2) = \delta(n-1).$$

Except for $t = 0$, $\phi_0(t)$ takes the value 0 on the half integers, and except for $t = \frac{1}{2}$ so does $\phi_1(t)$.

A version of the sampling theorem, for the multiwavelet case, is straightforward.

Let $\phi_0(t)$, $\phi_1(t)$ be cardinal orthogonal multiscaling functions. If $f(t) \in V_0(\phi_0, \phi_1)$, then

$$f(t) = \sum_n f(n)\,\phi_0(t-n) + f(n+1/2)\,\phi_1(t-n)$$

Shannon sampling using the sinc function can be expressed in this form using $\phi_0(t) = \mathrm{sinc}(2t)$, $\phi_1(t) = \mathrm{sinc}(2t-1)$.

The question becomes: do there exist cardinal orthogonal *multiscaling* functions $\phi_0(t)$, $\phi_1(t)$ of compact support and approximation order $K > 1$? Yes. In Section 5, examples of such functions will be given.

4. BALANCE ORDER

For traditional wavelet bases, the approximation order K is an important measure of how well the discrete-time wavelet transform (DWT) compresses smooth signals[1]. Indeed, for wavelet bases based on a single scaling function, the filter bank associated with the basis inherits the approximation properties of the basis. However, in the multiwavelet case, the situation is different. For multiwavelet bases, the filter bank does *not* inherit the approximation properties of the basis [11].

To be specific, the lowpass/highpass channels, of the filter bank associated with a traditional wavelet basis of approximation order K, preserve/annihilate the set $\mathcal{P}_{K-1}$ of polynomials of degree $k < K$. However, in the multiwavelet case, for the preservation/annihilation properties, it in not sufficient that the multiwavelet basis have approximation order[2] K. A stronger condition is required. Multiwavelet bases for which the zero moment properties *do* carry over to the discrete-time filter bank are called *balanced* after Lebrun and Vetterli [8, 9].

Specifically, multiwavelet bases for which the associated filter bank preserves/annihilates the set $\mathcal{P}_{K-1}$ of polynomials of degree $k < K$ are said to be *order-K balanced*. See [8, 9, 11, 12] for further details. For discrete-time signal processing, the order of balancing is more useful than the weaker order of approximation.

From [11], the condition for order-1 balancing for multiwavelet bases is

$$(z^{-3} + z^{-2} + z^{-1} + 1) \quad \text{divides} \quad H_0(z) + H_1(z). \quad (1)$$

Order-1 balanced multiwavelet filter banks preserve/annihilate constant signals. From [11], the condition for order-2 balancing is

$$(z^{-3} + z^{-2} + z^{-1} + 1)^2 \quad \text{divides} \quad H_0(z) + \left(\frac{3 - z^{-4}}{2}\right) H_1(z). \quad (2)$$

Order-2 balanced multiwavelets filter banks preserve/annihilate ramp and constant signals. The examples to be given in Section 5 will be balanced up to their approximation order.

5. CARDINAL MULTIWAVELET BASES

To obtain cardinal orthogonal multiscaling functions, it is useful to characterize them in terms of the scaling filters h_0 and h_1. For h_0, h_1 to generate orthogonal scaling functions ϕ_0, ϕ_1, it is necessary that h_0 and h_1 be orthogonal to their shifts by 4:

$$\sum_n h_i(n) h_j(n + 4k) = \delta(i - j) \cdot \delta(k) \quad (3)$$

The scaling functions ϕ_0, and ϕ_1 presented below are based on scaling filters h_0 and h_1 possessing a particular structure.

5.1. Order-2 Balanced Example

An order-2 balanced cardinal orthogonal system was obtained with scaling functions supported on $[0, 5]$ and scaling filters of length 11.

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[1]If $\int t^k \psi(t) dt = 0$ for $k = 0, \dots, K - 1$ (and not for $k = K$) then the wavelet basis is said to have approximation order K.

[2]If $\int t^k \psi_i(t) dt = 0$ for $i = 0, 1$ and $k = 0, \dots, K - 1$ (and not for $k = K$) then the multiwavelet basis is said to have approximation order K.

The scaling filters have the form

$$h_0(n) = \frac{1}{\sqrt{2}}(\quad a, \quad 0, \quad b, \quad 1, \quad c, \quad 0, \quad d, \quad 0, \quad e, \quad 0, \quad f)$$

$$(4)$$

$$h_1(n) = \frac{1}{\sqrt{2}}(-f, \quad 0, \quad e, \quad 0, -d, \quad 1, \quad c, \quad 0, -b, \quad 0, \quad a)$$

$$(5)$$

With this form, orthogonality between h_0 and h_1 is structurally incorporated. It is necessary only to choose the parameters so that h_0 is orthogonal to its own shifts by 4. The remaining free parameters will be used to attain balance order $K > 1$.

Our problem is to find $a, \dots, f$ such that h_0 and h_1 in (4,5) satisfy the orthogonality conditions (3) and the second order balancing conditions (2). This is a system of nonlinear equations — the balancing conditions (2) are linear, but the orthogonality conditions (3) are quadratic. The following solutions to this system of nonlinear equations were obtained using a lexical Gröbner basis [2] (for the computation of which, the software *Singular* was employed [7]).

$$A = -1/8 \pm \sqrt{15}/32$$
$$a = 1/32$$
$$b = A + 1/4$$
$$c = 15/16$$
$$d = -2A - 1/4$$
$$e = 1/32$$
$$f = A$$

As indicated, 2 solutions exist, however only one of them yields acceptable scaling functions, namely $A = -1/8 + \sqrt{15}/32$. That solution is shown in Figure 1. Note that ϕ_0, ϕ_1 shown in the figure are actually shifted cardinal functions, $\phi_0(3/2) = 1$ instead of $\phi_0(0) = 1$, etc.
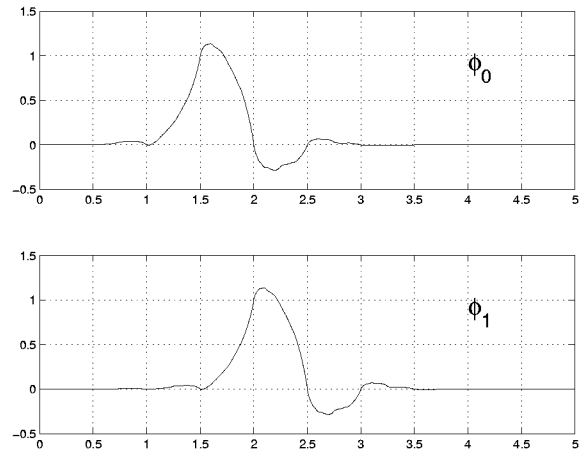

Figure 1: Order-2 balanced cardinal orthogonal scaling functions $\phi_0(t)$ and $\phi_1(t)$, with $A = -1/8 + \sqrt{15}/32$. The support of each is $[0, 5]$.

The wavelet filters h_2, h_3 are given by

$$h_2(n) = \frac{1}{\sqrt{2}} \left(-a, \quad 0, -b, \quad 1, -c, \quad 0, -d, \quad 0, -e, \quad 0, -f \right)$$

(6)

$$h_3(n) = \frac{1}{\sqrt{2}} \left(\quad f, \quad 0, -e, \quad 0, \quad d, \quad 1, -c, \quad 0, \quad b, \quad 0, -a \right),$$

(7)

for $n = 0, \ldots, 10$. All four analysis filters are obtained from the single prototype filter h_0. The special structure for h_0, h_1, h_2, h_3 guarantees orthogonality (3) provided that h_0 is orthogonal to its shifts by 4.

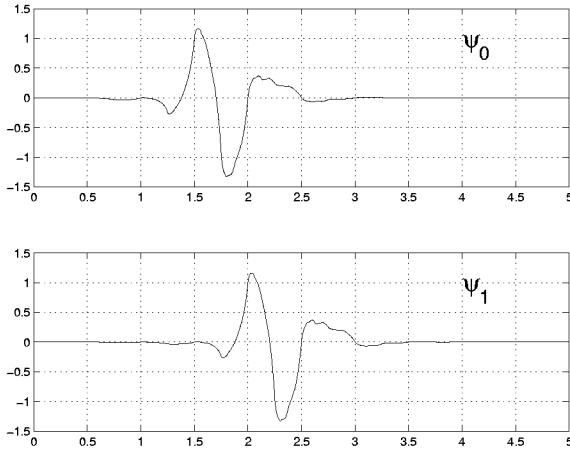The wavelets $\psi_0(t)$ and $\psi_1(t)$ are shown in Figure 2

Figure 2: Order-2 balanced wavelet functions $\psi_0(t)$ and $\psi_1(t)$, corresponding to the scaling functions shown in Figure 1. Like ϕ_0, ϕ_1, the wavelets ψ_0, ψ_1 are cardinal.

6. DISCUSSION

The use of cardinal wavelet bases also simplifies the initialization step of the discrete wavelet transform. That is the estimation of the the fine scale wavelet coefficients from the samples of a function — the estimation of $\int f(t)\phi(t - n)dt$ from $f(n)$. (See [13] for an overview of initialization methods.) However, with cardinal (or interpolating) scaling functions no such initialization step is needed. The samples $f(n)$ are themselves the values sought.

It must be noted that if a signal $f(t)$ lies in a scaling space $V(\phi)$ or $V(\phi_0, \phi_1)$, then generally there are translations $f(t - T)$ of the function that do not lie in the scaling space. Hence, in the multiresolution context there is a loss of shift-invariance, which occurs in both the wavelet and the multiwavelet cases. The requirement that a function and all its shifts lie in the same scaling space is very restrictive for sampling theorems, as discussed in [6].

7. CONCLUSION

The sampling issue has long been a concern in wavelets, both in theory and in practice. Obtaining wavelet coefficients from a sampled signal has previously required approximation or prefiltering.

However, with the new cardinal multiwavelet basis, interpolation and sampling issues are address without departing from orthogonal FIR multirate systems.

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