SEQUENTIAL EXTRACTION OF COMPONENTS OF MULTICOMPONENT PPS SIGNALS

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ABSTRACT

A procedure for parameter estimation of multicomponent Polynomial Phase Signals is presented. This scheme, while restricted to high SNR's, has the advantage of being extremely simple. It is also insensitive to the equal-coefficient identifiability problem of HAF (High-order Ambiguity Function) based methods. It poses, however, some restrictions on the component amplitudes. Its performance in noise is investigated, and confirmed with several examples.

1. INTRODUCTION

Polynomial-Phase Signals (PPS) have received a lot of attention, due to their importance in fields such as radar, sonar or communications. To estimate the parameters of these signals, three main types of approach have been proposed: tracking algorithms (e.g. [9]), algorithms based on the now called High-order Ambiguity Function (HAF) [7], and algorithms based on phase unwrapping and subsequent polynomial fitting [10], [2]. The approaches based on tracking are inherently extensive to multicomponent signals; the HAF based approaches have been satisfactorily extended to multicomponent PPS signals [1], [3]; the much simpler algorithms based on phase unwrapping remained limited to the monocomponent case.

In this paper, we will extend the phase unwrapping methods to the multicomponent case. An algorithm for sequential extraction of the individual components is presented. This "one component per cycle" type of behaviour is in contrast with the HAF based approaches, characterized by a "one order of exponents per cycle". While HAF has difficulties with components with the same highest order coefficients, this method will have difficulties with components of the same amplitude. Whenever possible, we will use continuous signal notation, even when referring to sampled signals. The paper is organized as follows. In Section 2, we will address the issue of unwrapping the phase of multicomponent PPS signals. In Section 3, the sequential extraction method is presented. In Section 4, the behaviour in noise is investigated, and compared to the Cramér-Rao bound by Monte-Carlo simulations.

2. PHASE UNWRAPPING OF MULTICOMPONENT COMPLEX SIGNALS

•Amplitude and Phase decomposition. To achieve uniqueness in the decomposition of observed (or computed) complex signals $c(t)$ into real amplitude and phase functions ($c(t) = b(t)\, e^{j\varphi(t)}$), several authors tend to force $b(t) \geq 0$. This choice of functions is perfectly in line with the mathematical view of the complex plane, but it may have adverse effects in the engineering field. If, for example, the sequence under analysis results from the heterodination of a sinusoid, $c(t) = \sin(\omega_1 t)\, e^{j\omega_2 t}$, imposing $b(t) \geq 0$ will give us a discontinuous phase function, with jumps of $\pm\pi$ whenever $t = n\pi/\omega_1$. To avoid these false discontinuities in the unwrapped phase, we should allow $b(t)$ to be negative. This means that, when unwrapping the phase of a complex sequence resulting from the heterodination of a bipolar signal, the uncertainty remaining after the $atan(\cdot)$ is one of a multiple of π, and not a multiple of 2π. There is another possible source of π discontinuities on the phase. Sometimes, the vector resulting from the sum of the individual components will pass trough the origin of the complex plane, forcing a jump of $\pm\pi$ in the overall phase. By considering a small perturbation in $c(t)$, we can easily see that these discontinuities are just the limiting process of a narrowing high derivative zone, and should thus be preserved in the unwrapped phase. To distinguish between the π discontinuities that should be preserved, and the ones that the unwrapping procedure should eliminate, we can use the estimate of $\varphi'(t)$:

$$\varphi'(t) = Im\left(\frac{c'(t)}{c(t)}\right). \qquad (1)$$

If a π discontinuity is, in fact, an attribute of the phase function, and not a false discontinuity created by $b(t)$, then $\varphi'(t)$ will reflect this, with a $\pi\delta(t)$ term. Otherwise, no such term will appear in $\varphi'(t)$. When using (1), we can run into numerical difficulties at points where $b(t) \approx 0$, but these can be handled by slightly perturbing $c(t)$. Since we will be dealing only with signals of the form

$$c(t) = \sum_{k=1}^{K} b_k\, e^{j\left(\sum_{m=0}^{M_k} a_{km} t^m\right)},$$

with no multiplicative bipolar signal, all π discontinuities will be features of the phase function, and the uncertainty will be reduced to multiples of 2π.

•Phase rate. Another source of discontinuities is the reduction to principal values done by the inverse trigonometric functions, which, in the case of complex sequences, leaves us with discontinuities of 2π radians. Eliminating

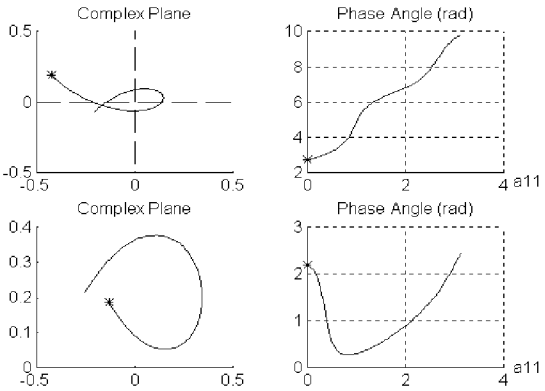

Figure 1: Phase locus for two component signals.

these discontinuities is not a trivial task. One of the main difficulties comes from the fact that the phase can change by more than 2π between samples, even for properly sampled signals. To see this, let us consider a simple signal, constituted by two complex sinusoids and a DC term:

$$c(n) = b_1 e^{j(a_{11}n+a_{10})} + b_2 e^{j(a_{21}n+a_{20})} + k. \qquad (2)$$

In the upper left plot of Figure 1, we represent in the complex plane the sample for $n = 0$ by an asterisk and the locus of all possible positions of the next sample by a solid line, assuming that the signal was correctly sampled. The values for the parameters were $b_1 = 1$, $b_2 = 1.5$, $0 \leq a_{11} < \pi$, $a_{21} = a_{11}/1.8$, $a_{10} = \pi/2$, $a_{20} = 3\pi/4$, and $k = 0.65\, e^{j\pi/15}$. The possible phase values for the second sample (as a function of a_{11}) are represented in the upper right plot.

As can be seen, even though the signal is properly sampled ($a_{21} < a_{11} < \pi$), the overall phase of the signal can increase more than 2π radians between samples. This means that phase jumps greater than 2π between successive samples can't be automatically associated with principal branch reduction, and may have to be left untouched in the unwrapped phase. It also means that between the phase angles of two successive samples we may have hidden multiples of 2π radians. Another disturbance comes from the fact that, even if all components have positive frequencies, the overall phase can decrease between samples, as can be seen in the bottom plots of Figure 1. We note that it has been pointed out that, for another class of signals, namely the unimodular analytic signals, the overall phase is guaranteed to be monotonically increasing [8]. As in the case of the π discontinuities, help comes from the fact that, by using (1), we can estimate $\varphi'(t)$ and, by integration, decide on the correct multiple of 2π. These two effects (rates of change greater than 2π rad/samples, and negative rates even for positive-frequencies-only signals), plus the existence of the necessary π discontinuities are very good reasons why $\varphi'(t)$ should never have been interpreted as being the Instantaneous Frequency of a complex signal (analytic or not) [4], [5].

•**The unwrapping algorithm.** The unwrapping algorithm to be used is thus very simple, and in the line of the one proposed in [11], the differences being due to the fact that we will be working in the time domain, and we will not perform the adaptive integration. (*i*) As a first step, we will estimate $\varphi'(t)$ with (1). To obtain reliable numerical derivatives of $c(t)$, a spectral procedure was devised, with proper care being taken to avoid aliasing. We may encounter some difficulties whenever the phase has a discontinuity, since the numerical derivative will try to approach a δ-function. We will thus limit the maximum amplitude of the derivative to a maximum of 2π rad/sample. (*ii*) The phase angle of $c(t)$ is obtained with a four quadrant $atan(\cdot)$ function. The phase of the first sample is considered properly unwrapped. (*iii*) An estimated unwrapped phase for the second sample is then computed, by projecting the unwrapped phase of the first sample with the local value of the derivative (assumed to be the arithmetic mean of the derivatives at the first and second sample). (*iv*) The unwrapped phase of the second sample is then obtained by simply adding (or subtracting) multiples of 2π to the principal value of its phase until agreement with the estimate within $\pm\pi$. Steps (*iii*) and (*iv*) are repeated for every sample.

3. SEQUENTIAL EXTRACTION OF COMPONENTS

The phase of multicomponent PPS signals has the interesting feature of being dominated by the phase of the strongest component. To appreciate this, let us consider a signal constituted by one fixed frequency component, and one linear chirp:

$$c(n) = b_1 e^{j\left(a_{12}n^2+a_{11}n+a_{10}\right)} + b_2 e^{j(a_{21}n+a_{20})} \qquad (3)$$

The unwrapped phase of this signal is represented in Figure 2, for the following set of parameters: $b_1 = 1.005$, $b_2 = 1$ (left plots), $b_1 = 1$, $b_2 = 1.005$ (right plots), $a_{12} = 0.17438$, $a_{11} = 0$, $a_{10} = 0$, $a_{21} = 6.1734 \times 10^{-2}$, $a_{20} = 0$. The phase of the individual components are also represented (dashed lines). Notice how the overall phase follows the phase of the strongest component, even though the amplitudes of the two components differ only in 0.5 %. Notice also how important it was to leave the π discontinuities in place, and how vital the estimate of $\varphi'(t)$ was to decide which way the π jumps took place. In this noise free and highly oversampled case, we could easily continue to correctly follow the strongest component up to ratios of amplitudes much closer to 1. The practical limit will, in the end, be defined by the decreasing width (and increasing height) of the peaks in their way to become δ functions, since they can be missed in the samples of $\varphi'(t)$. To further push this limit, we can interpolate $\varphi'(t)$, and then properly integrate it, instead of using the simpler mean derivative rule described above. For signals with more than two components, the phase of the strongest one will be followed if its amplitude is greater than the sum of the amplitudes of the other components. This selective behaviour of the overall phase allows us to estimate the parameters of the strongest PPS component, which will then be subtracted to the original data, and a new unwrapped phase obtained, for estimation of the second strongest component. The process will continue until no more components are left. The choice of the best criteria to decide when all components have been extracted is still being investigated. One promising option is to terminate

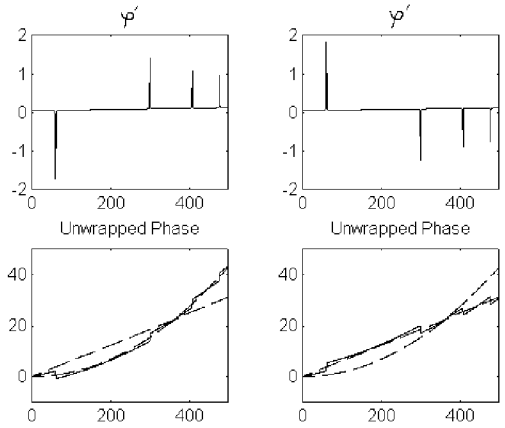

Figure 2: Unwrapped phase. Two component signal.

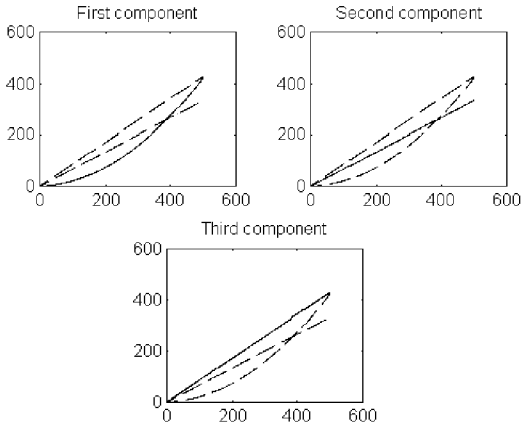

Figure 3: Three component signal.

the recursion when the estimated amplitude of a component falls below the treshold of uncertainty imposed by the observation noise.

As can be seen in Figure 2, the existence of other components will add interference to the unwrapped phase. If the observation interval is long compared to the inverse of the frequency difference between the components (this was not the case in Figure 2), this interference will be approximately zero mean, and can be dealt with by simply least squares fitting a polynomial of appropriate order to the unwrapped phase (not equivalent, in this context, to ML estimation). Some bias in the estimates will, however, remain. The choice of the order of the polynomial is best dealt with in the way suggested in [7] for the HAT: overestimate the order, and compare the highest order coefficient with the Crámer-Rao lower bound for the variance of that parameter. Decide that an order overestimate took place if the coefficient is small compared to the square root of the CRB. In practice, and for high SNR, the method seems to be very insensitive to overestimates of the order. Denoting by $\widehat{a}_{km}$ the estimated parameter for a_{km}, we thus have the following procedure: (i)Unwrap the phase; (ii)Obtain the estimates for the phase parameters of the strongest component by least squares fit to the unwrapped phase; (iii)Estimate the amplitude of the strongest component, by least squares fitting an exponential with the estimated phase to the signal; (iv)Subtract the estimated component from the signal, and start the procedure again, for the second component. After having estimated the individual components, we can isolate each one of them, in turn (by subtracting all the others from the original signal), and estimate its parameters again [3]. This new estimate should be less biased, since the interference of the other components has been greatly reduced. Convergence to the correct values is normally achieved after a few iterations.

As an example, let us consider (Figure 3) a three component signal, with $b_1 = 1$, $b_2 = 0.5$, $b_3 = 0.4$, $a_{12} = 0.0016$, $a_{11} = 0.05$, $a_{21} = 0.67$, $a_{31} = 0.86$, $a_{10} = a_{22} = a_{20} = a_{32} = a_{30} = 0$. In this case, one of the components crosses the other two in the time-frequency plane. This fact does not, however, hamper the performance of the extraction. The fit was done with third order polynomials, to show the negligible effects of order overestimation. After four iterations, the obtained estimates were $\widehat{a}_1 = [0,\ 0.0016,\ 0.0500,\ 0]$, $\widehat{a}_2 = [0,\ 0,\ 0.6700,\ 0]$, and $\widehat{a}_3 = [0,\ 0,\ 0.8600,\ 0]$.

4. PERFORMANCE IN NOISE

We can state our estimation problem as follows: we have a sequence with the following format:

$$\begin{aligned} c(n) &= b_1\, e^{j\left(\sum_{m=0}^{M_1} a_{1m} t^m\right)} + \\ &\quad + \sum_{k=2}^{K} b_k\, e^{j\left(\sum_{m=0}^{M_k} a_{km} t^m\right)} + w(n), \end{aligned}$$

where the components have been numbered by decreasing amplitude order, $b_n > \sum_{k=n+1}^{K} b_k$, and $w(n)$ is zero mean complex white Gaussian noise with variance σ_w^2. The SNR for each component is defined as $(b_k)^2/\sigma_w^2$. Assuming that we know M_1, we wish to estimate b_1 and the coefficients a_{1m}. To proceed, we can associate the noise term with the component to be estimated, and then use the result of [10], valid for high SNR, to transform it into an equivalent white Gaussian phase noise $v(n)$ with variance $\sigma_v^2 = \sigma_w^2/2(b_1)^2$. We will thus be left with the following model for the unwrapped phase of the first component:

$$\varphi_1(n) = \left(\sum_{m=0}^{M_1} a_{1m} t^m\right) + v(n) + i(n),$$

where $i(n)$ represents the (zero mean) interference on the unwrapped phase due to the joint effect of the other components. Simulation results show that, if there is only one interfering component and no unwrapping errors, $i(n)$ is approximately distributed with density $p(i) = \frac{1}{\pi}(L^2 - i^2)^{-1/2}$, for $-L < i < L$, where $L \approx \frac{b_2}{b_1}(1 - \frac{b_2}{b_1})^{-0.0872}$. If there are more interfering components in the signal, $i(n)$ will tend to become Gaussian distributed. The density function of the noise plus interference is thus dependent on the number of components. In the case of a single interfering component, its variance will be approximately equal

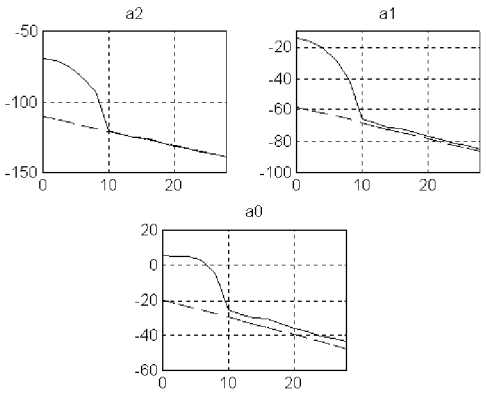

Figure 4: Variance of estimates. Monocomponent signal.

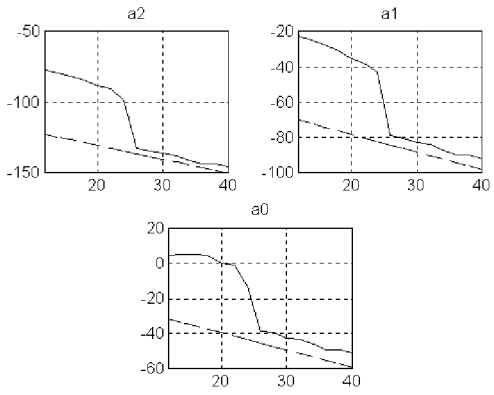

Figure 5: Variance of estimates. Three component signal.

to (the two terms are independent) $\sigma_v^2 + \sigma_i^2$, where $\sigma_i^2 = -\partial^2[J_0(\omega L)]/\partial\omega^2\big|_{\omega=0}$.

To evaluate the statistical efficiency of this estimator, the variances of the estimated phase parameters in the case of a single component signal were obtained by Monte Carlo simulation and compared against the asymptotic Cramér-Rao lower bounds [6] for several SNR ratios. Note that, in this single component case, $i(n) = 0$. As can be seen in Figure 4, the method approaches the CRB at medium to high SNR. The nonlinear treshold is at approximately 8 dB, which shows that, in this single component case, the unwrapping procedure performs similarly to the one proposed by Djúric and Kay in [2]. This was expected, since in both cases derivatives (finite differences, in the case of [2]) are being used for unwrapping.

In the multicomponent case, the contribution of $i(n)$ to the overall variance of the noise plus interference term will necessarily degrade the performance, and the Cramér-Rao bound (computed considering only the contribution of $w(n)$) will not be reachable. Worse then that is the fact that, when $\sum_{k=n+1}^{K} b_k$ approaches b_n, even a small contribution of noise may create unwrapping errors, which will render the results useless. In fact, the treshold region will move to higher and higher SNR's as $\sum_{k=n+1}^{K} b_k$ approaches b_n. Both effects can be seen in Figure 5, where the same three component signal of Figure 3 was used.

5. CONCLUSION

After proper unwrapping, the selective behaviour of the phase of a multicomponent PPS signal allows sequential estimation of the parameters of the individual components. The extraction/estimation is done one component at a time, while in the HAT based methods the sequence of estimation is done by order of exponent. This implies that, while HAT based methods have difficulties with multiple components with the same highest order coefficients, the present scheme has difficulties with multiple components of the same amplitude. In fact, for successful extraction, a hierarchical rule must be obeyed by the amplitudes of the several components. The method is limited to the high SNR region, and the required SNR will increase for increasingly similar amplitudes in the components.

6. REFERENCES

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