ROBUST SIGNAL DETECTION USING THE BOOTSTRAP

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ABSTRACT

This paper presents a CFAR detector based on the bootstrap for detecting signals with unknown amplitude, phase and frequency such as found in conventional pulsed radar and sonar systems. The detector is robust against non-Gaussian noise, and can still maintain the false alarm rate without much modification if consistent estimates are substituted for unknown parameters. Preliminary asymptotic results are given on the performance of the detector, and simulations are used to study the performance for small samples sizes.

1. INTRODUCTION

In this paper, we address the problem of signal detection in conventional pulsed radar or sonar systems. To deal with non-Gaussian interference, one approach in the literature is to use general parametric models [4]. Another approach is to use nonparametric techniques such as rank-based tests [2]. While this latter approach offers many advantages over parametric techniques, the great difficulty of the correct setting of the threshold to maintain a constant false alarm rate (CFAR) has remained largely unresolved. Typically, the threshold is set according to asymptotic results or derived through tedious Monte Carlo simulations and the detector does not perform well for the small-sample case.

We present here a nonparametric method based on the bootstrap [10] which is able to maintain a constant false alarm rate even for sample sizes as small as 100. The detector is noncoherent and is readily extended to the case where the signal frequency is unknown. A bootstrap detector for the deterministic signal case has already been reported in [9, 5], and although simulation results there indicate that its performance is not better than the CFAR matched filter's [8], they did show that the detector is robust against distributional deviations from the Gaussian noise assumption. In this contribution, the detector is for the signal with unknown parameters case and is attractive for the following reasons:

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- A constant false alarm rate is maintained without requiring the noise variance to be known. Hence, the detector is CFAR.
- The detector is robust against non-Gaussian noise such as noise with heavy-tailed distributions.
- Estimates of the unknown amplitude, frequency and phase are by-products of the detector. In addition, bootstrap confidence intervals for these estimators can be computed at little extra computational cost.
- When the frequency is known, amplitude-modulated signals can also be detected.

The paper is organised as follows. In the next section, the problem is formulated. Section 3 presents the detector based on the bootstrap. In section 4, the performance analysis is given. Simulation results and conclusions appear in sections 5 and 6 respectively.

2. PROBLEM FORMULATION

The problem of detecting a signal return can be expressed by the following model and hypotheses:

$$X_t = A\sin(\omega t + \phi) + W_t, \qquad t = 0, \ldots, N-1, \quad (1)$$

$$H_0 : A = 0 \quad \text{versus} \quad H_1 : A > 0, \quad (2)$$

where A, ω, ϕ are the unknown amplitude, frequency and phase of the signal with the following ranges $A \geq 0$, $\omega \in [0, \pi]$, $\phi \in [-\pi, \pi)$. The noise, $\{W_t\}$ are independent and identically-distributed noise with zero mean and variance σ^2.

When the frequency is known, and the noise distribution is known to be Gaussian, the non-CFAR optimum detector is applicable. Its test function for a nominal false alarm rate of α is [6],

$$\delta(\boldsymbol{x}) = \begin{cases} 1 & \\ \gamma & \text{if} \quad r \\ 0 & \end{cases} \begin{matrix} > \\ = \\ < \end{matrix} \sqrt{-N\sigma^2 \log(\alpha)}, \quad (3)$$

where $r = \sqrt{x_c^2 + x_s^2}$, $x_c = \sum_t x_t \cos(\omega t)$, and $x_s = \sum_t x_t \sin(\omega t)$, and $\boldsymbol{x} = (x_0, \ldots, x_{N-1})'$ are the observed samples. This test is derived by treating the phase as uniformly distributed in $[-\pi, \pi)$ in the likelihood ratio. Its probability of detection is given by

$$P_d = \int_\tau^\infty x e^{-(x^2+b^2)/2} I_0(bx)\, dx \triangleq Q(b, \tau), \qquad (4)$$

where I_0 is the zeroth-order modified Bessel function of the first kind, $\tau = \sqrt{-2\log(\alpha)}$ and $b^2 = NA^2/(2\sigma^2)$. $Q(b, \tau)$ is the so-called Marcum's Q function.

When the noise distribution is unknown, a better detector is the one based on least-squares regression and the F-test. The model in (1) is expressed as

$$X_t = \boldsymbol{h}_t' \boldsymbol{\beta} + W_t, \qquad t = 0, \ldots, N-1, \qquad (5)$$

where $\boldsymbol{\beta} = (A\cos\phi, A\sin\phi)'$, and $\boldsymbol{h}_t = (\sin(\omega t), \cos(\omega t))'$. The least-squares estimator of $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = (\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\boldsymbol{X}$, where $\mathbf{H} = (\boldsymbol{h}_0, \ldots, \boldsymbol{h}_{N-1})'$ and $\boldsymbol{X} = (X_1, \ldots, X_{N-1})'$. Under H_0, we test $\boldsymbol{\beta} = \mathbf{0}$ with the following test function [1],

$$\delta(\boldsymbol{x}) = \begin{cases} 1 & & > \\ \gamma & \text{if} \quad F & = \quad \tau, \\ 0 & & < \end{cases} \qquad (6)$$

where

$$F = \frac{N-2}{2} \frac{\hat{\boldsymbol{\beta}}'\mathbf{H}'\mathbf{H}\hat{\boldsymbol{\beta}}}{\boldsymbol{X}'(\mathbf{I} - \mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}')\boldsymbol{X}}, \qquad (7)$$

and τ is the threshold found from inverting the central F-distribution with 2 and $N-2$ degrees of freedom for a probability of $1-\alpha$.

Although robust against non-Gaussian noise, the F-test is not applicable when the frequency is replaced with an estimated frequency because the uncertainty in the frequency estimator is not taken into account by the test. In the next section, we propose a bootstrap-based detector that does automatically take this uncertainty into account when a frequency estimate is plugged in for the unknown frequency.

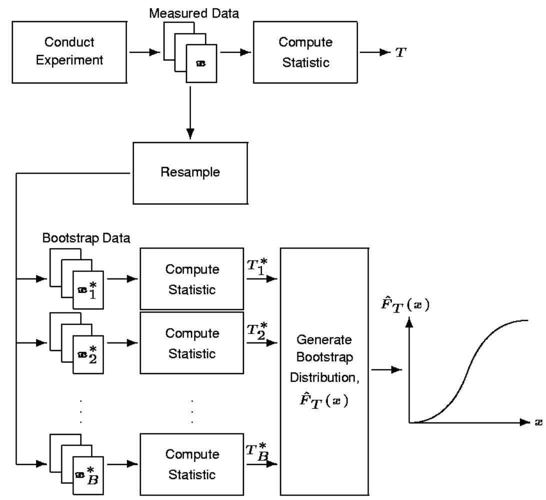
3. BOOTSTRAP-BASED DETECTOR

The bootstrap is a statistical method for estimating the sampling distribution of a statistic from the sample data itself (see Figure 1). In this way, modelling assumptions about the noise and signal are relaxed. It has been shown that, under some regularity conditions, bootstrap methods are second-order accurate as compared to the usual normal approximation [3], which is only first order accurate. For example, for a pivotal statistic T,

$$\Pr(T^* \le x) - \Pr(T \le x) = O_p(N^{-1}). \qquad (8)$$

The * indicates the bootstrap analogue, e.g., T^* is the statistic based on bootstrap (resampled) data $\boldsymbol{x}^*$ (see Figure 1).

Figure 1: The Bootstrap Procedure



3.1. Frequency Known

To derive a test for known frequency, we use the least-squares estimators of A and ϕ, which are $\hat{A} = \sqrt{\hat{\beta}_1^2 + \hat{\beta}_2^2}$ and $\hat{\phi} = \tan^{-1}(\hat{\beta}_2/\hat{\beta}_1)$, where $\hat{\beta}_1$ and $\hat{\beta}_2$ are the entries of $\hat{\boldsymbol{\beta}}$ given in the previous section. We then make the following conjecture.

$$\sqrt{N^c} \sup_x \left| \Pr\left(\hat{A}^*/\hat{\sigma}_{\hat{A}^*} \le x \middle| H_0\right) - \Pr\left(\hat{A}/\hat{\sigma}_{\hat{A}} \le x \middle| H_0\right) \right| \longrightarrow_{a.s.} 0, \qquad (9)$$

for some constant $c < 0$, and for consistent estimators, $\hat{\sigma}^2_{\hat{A}^*}$ and $\hat{\sigma}^2_{\hat{A}}$, of the variance of $\hat{A}^*$ and $\hat{A}$ respectively. This then motivates the following test function.

$$\delta(\boldsymbol{x}) = \begin{cases} 1 & & > \\ \gamma & \text{if} \quad \hat{A}/\hat{\sigma}_{\hat{A}} & = \quad \tau, \\ 0 & & < \end{cases} \qquad (10)$$

where τ is found from the bootstrap distribution of $\hat{A}^*/\hat{\sigma}_{\hat{A}^*}$, $\Pr\{\hat{A}^*/\hat{\sigma}_{\hat{A}^*} \le x | H_0\}$. The step-by-step procedure to compute τ is given in Table 1.

The variances, $\hat{\sigma}^2_{\hat{A}^*}$ and $\hat{\sigma}^2_{\hat{A}}$, are estimated using a nested bootstrap. Apply steps 3 and 4 on $\{x_t\}$ and each set of $\{x_t^*\}_b$, $b = 1, \ldots, B$, $t = 0, \ldots, N-1$, to obtain $\{\hat{A}^*_{b_1}\}$ and $\{\hat{A}^{**}_{b_1}\}_b$, $b = 1, \ldots, B$, $b_1 = 1, \ldots, B_1$. Then calculate, $\hat{\sigma}^2_{\hat{A}} = B_1^{-1}\sum_{b_1}(\hat{A}^*_{b_1} - B_1^{-1}\sum_{b_2}\hat{A}^*_{b_2})^2$ and $\hat{\sigma}^2_{\hat{A}^*} = B_1^{-1}\sum_{b_1}(\hat{A}^{**}_{b_1} - B_1^{-1}\sum_{b_2}\hat{A}^{**}_{b_2})^2$. Typical values for B and B_1 are 999 and 25 respectively. For more details on the nested bootstrap, see [10].

Table 1: Bootstrap procedure for computing the threshold.

Step 1. Compute the least-squares estimates, $\hat{A}$ and $\hat{\phi}$, from $x_0, \ldots, x_{N-1}$.

Step 2. Calculate the residuals, $\hat{w}_t = x_t - \hat{A}\sin(\omega t + \hat{\phi}), t = 0, \ldots, N-1$, and center to obtain $\tilde{w}_t = \hat{w}_t - N^{-1}\sum_k \hat{w}_k$.

Step 3. Resample residuals: $x_t^* = \tilde{w}_{k_t}$, where $\{k_t\}$ are generated from the discrete random variable K with probability $\Pr\{K = t\} = 1/N, t = 0, \ldots, N-1$.

Step 4. Compute the least-squares estimates, $\hat{A}^*$ and $\hat{\phi}^*$ from $x_0^*, \ldots, x_{N-1}^*$, and the bootstrap statistics, $T^* = \hat{A}^*/\hat{\sigma}_{\hat{A}^*}$.

Step 5. Repeat Steps 3–4 many times to obtain B bootstrap statistics, $T_1^*, \ldots, T_B^*$.

Step 6. Sort the B bootstrap statistics to get $T_{(1)}^* \leq \cdots \leq T_{(B)}^*$ and set the threshold, $\tau = T_{(q)}^*$, where $q = \lfloor (1-\alpha)(B+1) \rfloor$.

3.2. Frequency Unknown

When the frequency is not known, we substitute a consistent estimate $\hat{\omega}$ in its place. Table 1 applies with estimates $\hat{\omega}$ and $\hat{\omega}^*$ included in steps 1 and 4 respectively. This ease of incorporating unknowns into the detection scheme without needing a lot of modifications highlights one of the advantages of the bootstrap approach. Of course, however, the performance of the detector will depend on the accuracy of the estimators.

For estimating the frequency of a single tone, we investigated the use of $\hat{\omega} = \arg\max_\omega |d_X^{(N)}(\omega)|$ (where $d_X^{(N)}(\omega) = \sum_t x_t \exp(-j\omega t/N)$ is the finite Fourier transform), and a computationally faster method from [7] which also use $d_X^{(N)}(\omega)$. Estimators for A and ϕ are also conveniently found from the finite Fourier transform, $\hat{A} = \frac{2}{N}|d_X^{(N)}(\hat{\omega})|$ and $\hat{\phi} = \angle d_X^{(N)}(\hat{\omega}) + \pi/2$, without recoursing to least squares. A simulation study shows that these estimators are consistent although we do not report the results here.

4. PERFORMANCE ANALYSIS

As the performance of the bootstrap-based procedure is highly dependent on the estimators chosen as well as the noise distribution, closed-form expressions for the probability of false alarm and the probability of detection are not available. In this section, we state the asymptotic distribution of $T = \hat{A}/\hat{\sigma}_{\hat{A}}$ with only brief details. First, it can be shown by applying asymptotic theory of the finite Fourier transform [1], that the probability density of $\hat{A}$ is

$$f_{\hat{A}}(x) = \frac{x}{v}\exp(-(x^2 + A^2)/(2v))I_0(xA/v), \quad x \geq 0, \tag{11}$$

where $v = 2\sigma^2/N$. The variance of $\hat{A}$ estimated using the nested bootstrap step is asymptotically efficient and converges to $\sigma_{\hat{A}}^2 = (4-\pi)\sigma^2/N$ under H_0. The asymptotic density of $\hat{A}/\hat{\sigma}_{\hat{A}}$ is then

$$f_T(x) = ax\exp(-(ax^2 + A^2/v)/2)I_0(\sigma_{\hat{A}} xA/v), \; x \geq 0, \tag{12}$$

where $a = 2 - \pi/2$. The probability of detection is asymptotically given by $P_d = \int_\tau^\infty f_T(x)\,dx$, where τ is the threshold satisfying $\int_\tau^\infty f_T(x|A=0)\,dx = \alpha$.

5. SIMULATIONS RESULTS

In this section, we examine the performance of the detector and compare it with the classical ones based on uniformly-distributed phase and based on least-squares regression using simulations. Probabilities of false alarm and detection based on 100 simulations are tabulated in Tables 2 and 3 for the known frequency case with Gaussian and t_4 distributed noise respectively. Throughout the simulations $N = 100$, $A = 1$, $\phi = 1/4$, $\omega = 0.4\pi$, $\alpha = 0.05$, $B = 999$ and $B_1 = 25$. From the tables, it is seen that the bootstrap-based detector maintains the false alarm rate at the nominal level, although it is not as powerful as the other detectors. Tables 4 and 5 give the results for the unknown frequency case. Only the bootstrap-based detector is applicable in this case, highlighting once again the ease with which the bootstrap can handle plug-in estimates in place of unknowns with little modification required. The classical detectors do not work in this case even if the unknown frequency is replaced with an estimate.

6. CONCLUSIONS

This paper has presented a CFAR detector based on the bootstrap for signals with unknown amplitude, phase and frequency. The detector is robust against non-Gaussian noise, and can still maintain the false alarm rate without much modification if consistent estimates are substituted for unknown parameters. Preliminary asymptotic results were given on the performance of the detector, and simulations were used to study the performance for small samples of size 100.

Table 2: Estimated probabilities of false alarm (upper values) and detection (lower values) for known frequency and Gaussian noise. The detectors are based on the bootstrap (B), least-squares regression (F), and uniformly-distributed phase (Q).

	SNR (dB)				
Det.	-10	-5	0	5	10
B	0.01	0.09	0.03	0.05	0.06
	0.76	1.00	1.00	1.00	1.00
F	0.04	0.11	0.04	0.03	0.03
	0.82	1.00	1.00	1.00	1.00
Q	0.09	0.10	0.05	0.03	0.04
	0.80	1.00	1.00	1.00	1.00

Table 3: Estimated probabilities of false alarm (upper values) and detection (lower values) for known frequency and t_4 noise.

	SNR (dB)				
Det.	-10	-5	0	5	10
B	0.05	0.05	0.05	0.07	0.09
	0.77	0.98	1.00	1.00	1.00
F	0.05	0.07	0.09	0.04	0.04
	0.86	1.00	1.00	1.00	1.00
Q	0.07	0.08	0.07	0.04	0.02
	0.85	1.00	1.00	1.00	1.00

Table 4: Estimated probabilities of false alarm (upper values) and detection (lower values) for unknown frequency and Gaussian noise.

	SNR (dB)				
Det.	-10	-5	0	5	10
B	0.05	0.05	0.05	0.04	0.05
	0.17	0.71	1.00	1.00	1.00

Table 5: Estimated probabilities of false alarm (upper values) and detection (lower values) for unknown frequency and t_4 noise.

	SNR (dB)				
Det.	-10	-5	0	5	10
B	0.05	0.03	0.01	0.02	0.00
	0.18	0.71	0.99	1.00	1.00

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