OPTIMAL GENERALIZED SAMPLING EXPANSION

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ABSTRACT

This work presents an analysis of Papoulis' Generalized Sampling Expansion (GSE) for a wide-sense stationary signal with a known power spectrum in the presence of quantization noise. We find the necessary and sufficient conditions for a GSE system to produce the minimum mean squared error while using the optimal linear estimation filter. This is actually an extension of the optimal linear equalizer (linear source/channel optimization) to the case of M parallel channels.

1. INTRODUCTION

In his famous Generalized Sampling Expansion (GSE) [1],[2] Papoulis has shown the following: A signal $f(t)$ of finite energy, bandlimited to $\omega \in [-B, B]$, passing through M LTI systems and generating responses $g_k(t), k = 1, \ldots, M$, can be uniquely reconstructed, under some conditions on the M filters, from the samples of the output signals $g_k(nT)$, sampled at $1/M$ the Nyquist rate. While Nyquist samples are π/B apart, the outputs in a GSE system are sampled at $T = M\pi/B$ apart. Papoulis found the explicit reconstruction formula of $f(t)$ from the samples $g_k(nT)$. Another version of the GSE was introduced by Brown [3]. The analysis conducted by Papoulis and Brown assumed knowledge of the exact values of the samples. In practice, we never have the exact values of the samples due to quantization and noise. Therefore the question of how well is the reconstruction in the presence of noise immediately arises. Cheung and Marks [4] were the first to discuss these issues for deterministic input signals. They found a sufficient condition for Ill-Posedness, where under their definition, an ill-posed GSE system is a system which produces an un-bounded reconstruction error when a small amount of noise is imposed on the samples. Brown and Cabrera [5],[6], found a necessary and sufficient condition for well-Posedness. Under the assumption of white quantization noise, Seidner and Feder [7] found the necessary and sufficient conditions for the best possible GSE system, in the sense of producing the minimal mean squared reconstruction error.

For a single sampling channel having one input and one output, where the input is a wide-sense stationary (WSS) signal with a known power spectrum, optimal filtering analysis in the presence of additive noise, has been widely explored (see e.g. [8],[9],[10],[11],[12]). This is actually the well-known problem of optimal linear equalization (linear combined source/channel analysis). Costas [13] found the optimal pre-filter and post-filter required to produce the minimal mean squared reconstruction error of a channel with additive noise. For the case of a bandlimited WSS input signal sampled at the Nyquist rate, his result leads to the optimal pre-filter which should be used prior to the sampling followed by the post-filter (multiplied by a factor of π/B due to the sampling).

Although mentioned by Kahn and Liu [14], a similar analysis for GSE system, i.e. for more than one sampling channel, does not appear in literature.

We here consider the sampling of M channels resulting from feeding a WSS signal $x(t)$ with a known power spectrum $S_{xx}(\omega)$ into M filters $H_k(\omega)$ and adding different stochastic noise sequences to the M sampled signals as an M dimensional communication channel. We find the optimal filters $H_k(\omega)$ where $k = 1, \ldots, M$ which are the solution for the optimal linear equalization problem (or the linear solution of the combined source/channel problem) for an M-dimensional channel. These filters represent the optimal GSE system.

2. PROBLEM FORMULATION

The transfer function of a GSE system is denoted $\mathbf{H}(\omega)$, where $\mathbf{H}(\omega)$ is an M dimensional vector, and so we have

$$\mathbf{G}(\omega) = \mathbf{H}(\omega)X(\omega) \tag{1}$$

where $\mathbf{G}(\omega)^T = [G_1(\omega), \ldots, G_M(\omega)]$ and $X(\omega), G_k(\omega)$ are the Fourier transforms of the input signal $x(t)$ and the k-th output channel $g_k(t)$ respectively. When sampling at $1/M$ the Nyquist rate, i.e. at a sampling period $T = M\pi/B$, we get aliased versions of the output signals, which, at the frequency domain, are periodic with a period $c = 2B/M$. We denote by $G_k^a(\omega)$ the Fourier transform of the sampled k-th output signal, and observe that since it is periodic with a period c it is sufficient to consider only one period, say $\omega \in [-B, -B+c]$. At this region, $G_k^a(\omega)$ is composed of M replicas of $G_k(\omega)$, the Fourier transform of the k-th output signal, shifted in frequency by multiples of c, i.e.

$$G_k^a(\omega) = \left(\frac{c}{2\pi}\right) \sum_{i=0}^{M-1} G_k(\omega + ic), \qquad \omega \in [-B, -B+c] \tag{2}$$

Since $G_k(\omega) = H_k(\omega)X(\omega)$ where $H_k(\omega)$ is the k-th component of the LTI system transfer vector $\mathbf{H}(\omega)$, we have

$$G_k^a(\omega) = \left(\frac{c}{2\pi}\right) \sum_{i=0}^{M-1} H_k(\omega+ic)X(\omega+ic), \omega \in [-B, -B+c] \tag{3}$$

This is true for $k = 1, 2, \ldots, M$, and so we may write, in a matrix form:

$$\mathbf{G}^a(\omega) = \left(\frac{c}{2\pi}\right) \mathbf{T}(\omega)\mathbf{X}^a(\omega) \quad \omega \in [-B, -B+c] \tag{4}$$

where $\mathbf{G}^a(\omega)^T = [G_1^a(\omega), G_2^a(\omega), \ldots, G_M^a(\omega)]$ and

$$\mathbf{X}^a(\omega)^T = [X(\omega), X(\omega+c), \ldots, X(\omega+(M-1)c)] \tag{5}$$

i.e. its l-th component is $X(\omega + (l-1)c)$, and finally, $\mathbf{T}(\omega)$ is an $M \times M$ matrix whose (k,l)-th component is given by

$$T_{kl}(\omega) = H_k(\omega + (l-1)\cdot c) \tag{6}$$

We assume now that $x(t)$ is a bandlimited WSS signal with a known bandlimited power spectrum $S_{xx}(\omega)$ in $\omega \in [-B, B]$ and that every two frequency components of the signal are uncorrelated. Although $x(t)$ does not have a Fourier transform since it has infinite energy, $\lim_{\tau\to\infty} E\left\{|X(\omega)_\tau|^2\right\}$ exists for WSS signals, where

$$X(\omega)_\tau = \frac{1}{\sqrt{2\tau}} \int_{-\tau}^{\tau} x(t)e^{j\omega t}dt \tag{7}$$

Since our concern is with such expectations we will replace $X(\omega)$ with $X(\omega)_\tau$ when expectations are calculated.

Adding a zero mean discrete stochastic noise sequence to each of the samples sequences resulting from sampling the M output channels of a GSE system prior to reconstruction, is equal to adding a bandlimited WSS noise signal $v_k(t)$ to the k-th channel prior to the sampling where $S_{v_k v_k}(\omega)$ exists for $\omega \in [-B, -B+c]$. In the frequency domain this is done by adding the vector $(c/2\pi)\mathbf{V}^a(\omega)$ to $\mathbf{G}^a(\omega)$ where $\mathbf{V}^a(\omega)^T = [V_1(\omega), \ldots, V_M(\omega)]$ and $V_k(\omega)$ is the (non-existing) Fourier transform of $v_k(t)$ prior to the sampling.

Thus the model describing the system is

$$\mathbf{Y}(\omega) = \left(\frac{c}{2\pi}\right) \mathbf{T}(\omega)\mathbf{X}^a(\omega) + \left(\frac{c}{2\pi}\right) \mathbf{V}(\omega) \tag{8}$$

The correlation matrix $\mathbf{C_{xx}}(\omega)_\tau$ is

$$\mathbf{C_{xx}}(\omega)_\tau = E\left\{\mathbf{X}^a(\omega)_\tau \mathbf{X}^a(\omega)_\tau^{T^*}\right\} \tag{9}$$

It is well known, (see e.g. [2]), that

$$\mathbf{C_{xx}}(\omega) = \lim_{\tau\to\infty} \mathbf{C_{xx}}(\omega)_\tau = \mathbf{S_{xx}}(\omega) \tag{10}$$

where $\mathbf{S_{xx}}(\omega)$ is the cross-spectrum matrix of the vector $\mathbf{X}^a(\omega)$:

$$\mathbf{S_{xx}}(\omega) = diag\left\{S_{xx}(\omega), S_{xx}(\omega+c), \ldots, S_{xx}(\omega+(M-1)c)\right\} \tag{11}$$

Similarly we find

$$\mathbf{C_{xy}}(\omega) = \left(\frac{c}{2\pi}\right) \mathbf{S_{xx}}(\omega)\mathbf{T}(\omega)^{T^*} \tag{12}$$

and

$$\mathbf{C_{yy}}(\omega) = \left(\frac{c}{2\pi}\right)^2 \mathbf{T}(\omega)\mathbf{S_{xx}}(\omega)\mathbf{T}(\omega)^{T^*} + \left(\frac{c}{2\pi}\right)^2 \mathbf{S_{vv}}(\omega) \tag{13}$$

where $\mathbf{S_{vv}}(\omega)$ is the cross-spectrum matrix of the vector $\mathbf{V}(\omega)$, i.e. $\mathbf{S_{vv}}(\omega) = diag\left\{S_{v_1v_1}(\omega), \ldots, S_{v_Mv_M}(\omega)\right\}$.

Using the well known Bayesian estimation analysis (see Kay [15] for example) we find the reconstruction formula

$$\hat{\mathbf{X}}^a(\omega) = \mathbf{P}(\omega)\mathbf{Y}(\omega) \tag{14}$$

where the (l,k)-th component of $\mathbf{P}(\omega)$ is the l-th frequency slice of the k-th reconstruction filter, i.e. $P_k(\omega + (l-1)c)$. Thus, the matrix $\mathbf{P}(\omega)$ fully represents the M reconstruction filters $P_k(\omega)$. $\mathbf{P}(\omega)$ is found by

$$\begin{aligned} \mathbf{P}(\omega) &= \mathbf{C_{xy}}(\omega)\mathbf{C_{yy}}(\omega)^{-1} \\ &= \left(\frac{2\pi}{c}\right) \mathbf{S_{xx}}(\omega)\mathbf{T}(\omega)^{T^*} \times \\ &\quad \left[\mathbf{T}(\omega)\mathbf{S_{xx}}(\omega)\mathbf{T}(\omega)^{T^*} + \mathbf{S_{vv}}(\omega)\right]^{-1} \end{aligned} \tag{15}$$

In this case of a linear model we find that the cross-spectrum matrix of the reconstruction error $\mathbf{e}^a(\omega) = \hat{\mathbf{X}}^a(\omega) - \mathbf{X}^a(\omega)$ is

$$\mathbf{S_{ee}}(\omega) = \left[\mathbf{S_{xx}}(\omega)^{-1} + \mathbf{T}(\omega)^{T^*}\mathbf{S_{vv}}(\omega)^{-1}\mathbf{T}(\omega)\right]^{-1} \tag{16}$$

In order to have a meaningful optimal GSE system we need to enforce a power constraint on the average power sent to the channel. We have chosen the following constraint

$$\sigma_x^2 = \frac{1}{2\pi} \int_{-B}^{-B+c} trace\left\{\mathbf{T}(\omega)\mathbf{S_{xx}}(\omega)\mathbf{T}(\omega)^{T^*}\right\} d\omega \tag{17}$$

where $\sigma_x^2 = E\left\{|x(t)|^2\right\}$. This means that the total power of the sum of the M outputs (prior to sampling) is equal to the input signal power.

We now want to find the system $\mathbf{T}(\omega)$ which minimizes the mean squared reconstruction error subject to the power constraint. The functional to be minimized is

$$J = \frac{1}{2\pi}\int_{-B}^{-B+c} f\left(\mathbf{T}(\omega),\lambda\right) d\omega \tag{18}$$

where

$$f\left(\mathbf{T}(\omega),\lambda\right) = trace\left\{\mathbf{S_{ee}}(\omega) + \lambda\mathbf{T}(\omega)\mathbf{S_{xx}}(\omega)\mathbf{T}(\omega)^{T^*}\right\} \tag{19}$$

where the first term above is actually an integral on the power spectrum of the error, the second term results from the power constraint and λ is the Lagrange multiplier.

3. THE OPTIMAL GSE SYSTEM

The solution is found by taking the derivative of J with respect to $\mathbf{T}(\omega)$ treating the (i,j)-th component of $\mathbf{T}(\omega)$, i.e. $T_{ij}(\omega)$, and its conjugate $T_{ij}^*(\omega)$ as independent variables (See Appendix A in Therrien [16]). Thus, the optimal $\mathbf{T}(\omega)$ satisfies $\frac{\partial f}{\partial T_{ij}(\omega)} = 0$ and $\frac{\partial f}{\partial T_{ij}^*(\omega)} = 0$ for $i = 1,\ldots,M$ and $j = 1,\ldots,M$ yielding

$$\lambda\left[\mathbf{S_{xx}}(\omega)^{-1} + \mathbf{T}(\omega)^{T^*}\mathbf{S_{vv}}(\omega)^{-1}\mathbf{T}(\omega)\right]^2 = \mathbf{T}(\omega)^{T^*}\mathbf{S_{vv}}(\omega)^{-1}\mathbf{T}(\omega)^{T^{*-1}}\mathbf{S_{xx}}(\omega)^{-1} \tag{20}$$

Let us first discuss the simple case of a diagonal $\mathbf{T}(\omega)$. The solution for this particular case is given by

$$\mathbf{T}(\omega)^{T^*}\mathbf{T}(\omega) = \frac{1}{\sqrt{\lambda}}\mathbf{S_{vv}}(\omega)^{1/2}\mathbf{S_{xx}}(\omega)^{-1/2} - \mathbf{S_{vv}}(\omega)\mathbf{S_{xx}}(\omega)^{-1} \tag{21}$$

where λ is found from the power constraint, i.e. from equation (17). This is actually a case of M disjoint sub-band channels which is identical to the one channel case, where $S_{vv}(\omega)$ of the single channel with $\omega \in [-B, B]$ is split into M components $S_{vv}(\omega + (k-1)c)$ with $\omega \in [-B, -B+c]$ which are used as the components of $\mathbf{S_{vv}}(\omega)$. We then get similar equations for both the GSE and the single channel cases. Note that the right-hand side of equation (21) is a diagonal matrix. We denote this matrix by $\mathbf{D}(\omega)$. This matrix must be positive semi-definite matrix. For all frequencies for which one of the components of $\mathbf{D}(\omega)$ is negative, it must be replaced by 0. This means that for these frequencies the appropriate component of $\mathbf{T}(\omega)$ must be zero.

We now discuss the case where $\mathbf{S_{vv}}(\omega) = S_{vv}(\omega)\mathbf{I}$, i.e. the quantization noise of all M channels has the same power spectrum $S_{vv}(\omega)$ for $\omega \in [-B, -B+c]$. It is easy to see that the solution is also given by equation (21). Note again that for all frequencies for which one of the components of $\mathbf{D}(\omega)$ is negative, it must be replaced by 0. For these frequencies the appropriate column of $\mathbf{T}(\omega)$ must be a zero vector.

If all the components along the main diagonal of $\mathbf{D}(\omega)$ are distinct for every ω, then by using singular value decomposition of $\mathbf{T}(\omega)$, i.e. $\mathbf{T}(\omega) = \mathbf{A}(\omega)\mathbf{\Sigma}(\omega)\mathbf{B}(\omega)^{T^*}$, it is easy to see that

$$\mathbf{T}(\omega) = \mathbf{U}(\omega)\mathbf{\Sigma}(\omega)_i\mathbf{B}(\omega)_i^{T^*} \tag{22}$$

where $\mathbf{U}(\omega)$ is an arbitrary unitary matrix, $\mathbf{B}(\omega)_i$ is a column permutation of $diag\left\{e^{j\alpha_1(\omega)},\ldots,e^{j\alpha_M(\omega)}\right\}$ where $\alpha_k(\omega)$ is an arbitrary function of ω and $\mathbf{\Sigma}(\omega)_i$ is a similar permutation of the square root of $\mathbf{D}(\omega)$. Note that we may choose a different permutation for every ω.

If some k components of $\mathbf{D}(\omega)$ are equal, we may choose any $k \times k$ unitary matrix and substitute its components with the corresponding components of $\mathbf{B}(\omega)$ (i.e. in the the intersections of the k rows and columns of the non-distinct values of $\mathbf{D}(\omega)$).

An explicit expression for $\mathbf{T}(\omega)$ of the general case has not been found yet.

4. EXAMPLES

As the first example we discuss the case of white signal and white noise. We denote the n-th noise value which is added to the n-th sample of the k-th channel as $v_k(nT)$ where $E\{v_k(nT)v_q^*(mT)\} = \sigma_v^2\,\delta_{k,q}\delta_{n,m}$ and $v_k(nT)$ is a zero mean random variable. We may therefore consider the sequence $v_k(nT)$ as a sample sequence of a white WSS stochastic process $v_k(t)$, which is bandlimited to $\omega \in [-B, -B+c]$ where $c = 2B/M$, and has a spectral power density of $S_{vv}(\omega) = N_v$ (prior to sampling). Similarly, the signal $x(t)$ is a bandlimited white noise with $S_{xx}(\omega) = N_x$ for $\omega \in [-B, B]$. Therefore we have $\mathbf{S_{xx}}(\omega) = N_x\mathbf{I}$ and $\mathbf{S_{vv}}(\omega) = N_v\mathbf{I}$, For this case we find that the optimal GSE system satisfies $\mathbf{T}(\omega)^{T^*}\mathbf{T}(\omega) = \mathbf{I}$ for $\omega \in [-B, -B+c]$ and that $\mathbf{P}(\omega) = \left(\frac{2\pi}{c}\right)\mathbf{T}(\omega)^{-1}\left[\frac{N_x}{N_x+N_v}\right]$. This is equal to the optimal $\mathbf{T}(\omega)$ for the deterministic case which is described in [7]. (A factor of M is required there because of a different power constraint).

The second example is a GSE system in which $S_{v_kv_k}(\omega) = \frac{1}{M}\sum_{i=1}^{M} S_{vv}(\omega + (i-1)c)$ for $\omega \in [-B, -B+c]$ where $S_{vv}(\omega)$ is the noise power spectrum of a single channel. Thus, $S_{v_kv_k}(\omega)$ is an aliased version of the original noise. We assume that the noise sequences of different channels are still uncorrelated. We now find the optimal equalizer $H_{opt}(\omega)$ for a single channel having noise spectrum which is $S_{v_kv_k}(\omega)$ duplicated M times into $\omega \in [-B, B]$. When we choose:

$T_{k,l}(\omega) = e^{j\omega(k-1)(l-1)2\pi/M} H_{opt}(\omega + (l-1)c)/\sqrt{M}$ it is easy to find particular cases of $S_{xx}(\omega)$ and $S_{vv}(\omega)$ for which this system produces a mean squared error smaller than this of the optimal original single channel which had the original $S_{vv}(\omega)$. (Note that this system satisfies equation (20)). This implies that such GSE systems may produce a smaller reconstruction error then a single channel.

5. CONCLUSION

We have found the equation of the optimal filter bank equalizer for the case of sampling of a bandlimited WSS input signal with an additive bandlimited WSS noise. We had also shown that a GSE system may outperform a single channel although the explicit formula of the optimal GSE system has not been found for the general case.

6. REFERENCES

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