MINIMAX ROBUST TIME-FREQUENCY FILTERS FOR NONSTATIONARY SIGNAL ESTIMATION*

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ABSTRACT

We introduce minimax robust time-varying Wiener filters and show a result that facilitates their calculation. Reformulation in the time-frequency domain yields simple closed-form expressions of *minimax robust time-frequency Wiener filters* based on three different uncertainty models. For one of these filters, an efficient implementation using the multi-window Gabor transform is proposed.

1 INTRODUCTION

We consider the estimation of a nonstationary random signal $s(t)$ from an observation $r(t) = s(t) + n(t)$, where $n(t)$ is nonstationary noise uncorrelated with $s(t)$, by means of a linear, time-varying system $\mathbf{H}$. The resulting mean square error (MSE) $e(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n) \triangleq \mathrm{E}\{\|\mathbf{H}r - s\|_2^2\}$ is given by[1]

$$e(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n) = \mathrm{tr}\big\{(\mathbf{I}-\mathbf{H})\mathbf{R}_s(\mathbf{I}-\mathbf{H})^+ + \mathbf{H}\mathbf{R}_n\mathbf{H}^+\big\}. \quad (1)$$

The MSE is minimized by the *time-varying Wiener filter* [1]

$$\mathbf{H}_W \triangleq \arg\min_{\mathbf{H}} e(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n) = \mathbf{R}_s(\mathbf{R}_s+\mathbf{R}_n)^{-1}, \quad (2)$$

and the minimal MSE can be expressed as

$$e_{\min}(\mathbf{R}_s,\mathbf{R}_n) \triangleq e(\mathbf{H}_W;\mathbf{R}_s,\mathbf{R}_n) = \mathrm{tr}\big\{\mathbf{R}_s(\mathbf{R}_s+\mathbf{R}_n)^{-1}\mathbf{R}_n\big\}. \quad (3)$$

The Wiener filter's sensitivity to deviations of the actual correlations from the nominal correlations motivates the use of *minimax robust Wiener filters*. This paper extends the robust Wiener filters proposed in [2]–[5] for *stationary* processes to the nonstationary case (see also [6, 7]). Complementing the introduction of robust time-varying Wiener filters in [8], Section 2 provides a fundamental result that facilitates the calculation of such filters. A further simplification is achieved in Section 3 by a time-frequency formulation. Explicit expressions of "minimax robust time-frequency Wiener filters" are derived for three uncertainty models. Finally, simulation results are presented in Section 4.

2 ROBUST TIME-VARYING WIENER FILTER

By definition, the *minimax robust time-varying Wiener filter* $\mathbf{H}_R$ optimizes the worst-case performance within *uncertainty classes* $\mathcal{S}$, $\mathcal{N}$ for the correlations $\mathbf{R}_s$, $\mathbf{R}_n$:

$$\mathbf{H}_R \triangleq \arg\min_{\mathbf{H}} \max_{\substack{\mathbf{R}_s\in\mathcal{S}\\ \mathbf{R}_n\in\mathcal{N}}} e(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n). \quad (4)$$

*Funding by FWF grant P11904-TEC.

[1] Here, $\mathbf{R}_s$ and $\mathbf{R}_n$ denote the correlation operators of $s(t)$ and $n(t)$, respectively. The correlation operator $\mathbf{R}_x$ of a (generally nonstationary) random process $x(t)$ is the positive (semi-)definite linear operator whose kernel equals $r_x(t,t') = \mathrm{E}\{x(t)\,x^*(t')\}$. In a discrete-time setting, $\mathbf{R}_x$ would be a matrix.

The uncertainty classes $\mathcal{S}$, $\mathcal{N}$ model our uncertainty about the actual correlations. All $\mathbf{R}_s \in \mathcal{S}$ are assumed to have the same trace (mean energy of $s(t)$) $\bar{E}_s \triangleq \mathrm{E}\{\|s\|_2^2\} = \mathrm{tr}\{\mathbf{R}_s\}$, and similarly for $\mathbf{R}_n \in \mathcal{N}$.

The calculation of $\mathbf{H}_R$ simplifies if

$$\min_{\mathbf{H}} \max_{\substack{\mathbf{R}_s\in\mathcal{S}\\ \mathbf{R}_n\in\mathcal{N}}} e(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n) = \max_{\substack{\mathbf{R}_s\in\mathcal{S}\\ \mathbf{R}_n\in\mathcal{N}}} \min_{\mathbf{H}} e(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n), \quad (5)$$

since $\min_{\mathbf{H}} e(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n)$ is achieved by the ordinary Wiener filter $\mathbf{H}_W = \mathbf{R}_s(\mathbf{R}_s+\mathbf{R}_n)^{-1}$ in (2). Hence, when (5) is valid, $\mathbf{H}_R$ is equal to the *ordinary* Wiener filter

$$\mathbf{H}_R = \mathbf{H}_W^L \triangleq \mathbf{R}_s^L\big(\mathbf{R}_s^L+\mathbf{R}_n^L\big)^{-1}$$

obtained for those correlations $\mathbf{R}_s^L$, $\mathbf{R}_n^L$ that are *least favorable* in the sense that they maximize $e_{\min}(\mathbf{R}_s,\mathbf{R}_n) = \min_{\mathbf{H}} e(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n)$ among all $\mathbf{R}_s \in \mathcal{S}$ and $\mathbf{R}_n \in \mathcal{N}$, i.e.,

$$(\mathbf{R}_s^L,\mathbf{R}_n^L) = \arg\max_{\substack{\mathbf{R}_s\in\mathcal{S}\\ \mathbf{R}_n\in\mathcal{N}}} e_{\min}(\mathbf{R}_s,\mathbf{R}_n), \quad (6)$$

with $e_{\min}(\mathbf{R}_s,\mathbf{R}_n)$ given by (3).

It can be shown [9] that the pivotal relation (5) holds if and only if there exists a *saddle point* of $e(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n)$, i.e., a filter $\mathbf{H}_L$ and correlations $\mathbf{R}_s^L$, $\mathbf{R}_n^L$ satisfying

$$e(\mathbf{H}_L;\mathbf{R}_s,\mathbf{R}_n) \le e(\mathbf{H}_L;\mathbf{R}_s^L,\mathbf{R}_n^L) \le e(\mathbf{H};\mathbf{R}_s^L,\mathbf{R}_n^L) \quad (7)$$

for all $\mathbf{H}$ and $\mathbf{R}_s \in \mathcal{S}$, $\mathbf{R}_n \in \mathcal{N}$. The right-hand inequality in (7) is trivially satisfied by choosing $\mathbf{H}_L = \mathbf{H}_W^L$ since $\mathbf{H}_W^L$ minimizes $e(\mathbf{H};\mathbf{R}_s^L,\mathbf{R}_n^L)$. A necessary and suffient condition for the left-hand inequality in (7) is provided by the following theorem whose proof is outlined in the Appendix.

Theorem 2.1. *For convex*[2] *uncertainty classes* $\mathcal{S}$, $\mathcal{N}$, *there is* $e(\mathbf{H}_W^L;\mathbf{R}_s,\mathbf{R}_n) \le e(\mathbf{H}_W^L;\mathbf{R}_s^L,\mathbf{R}_n^L)$ *with* $\mathbf{H}_W^L = \mathbf{R}_s^L\big(\mathbf{R}_s^L+\mathbf{R}_n^L\big)^{-1}$ *if and only if* $\mathbf{R}_s^L$ *and* $\mathbf{R}_n^L$ *are least favorable correlations as defined in* (6).

Hence, we have finally simplified the calculation of $\mathbf{H}_R$ to the convex optimization problem (6).

3 ROBUST TIME-FREQUENCY WIENER FILTER

A further simplification will be achieved by a time-frequency (TF) reformulation in terms of the Weyl symbol $L_{\mathbf{H}}(t,f)$ of a linear time-varying system $\mathbf{H}$ [10]–[12] and the Wigner-Ville spectrum (WVS) $\overline{W}_x(t,f)$ of a nonstationary random process $x(t)$ [13]–[15]. This will allow us to replace the calculus of operators by the simpler calculus of functions. We

[2] A set $\mathcal{S}$ is convex if from $\mathbf{R}_1 \in \mathcal{S}$ and $\mathbf{R}_2 \in \mathcal{S}$ it follows that $\alpha\mathbf{R}_1 + (1-\alpha)\mathbf{R}_2 \in \mathcal{S}$ for $0 \le \alpha \le 1$.

require the processes $s(t)$ and $n(t)$ to be jointly underspread [15], i.e., to feature only a limited amount of TF correlation. For underspread processes, the following approximate TF formulations[3] of $e(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n)$ in (1), $\mathbf{H}_W$ in (2), and $e_{\min}(\mathbf{R}_s,\mathbf{R}_n)$ in (3) can be derived [16],

$$e(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n) \;\approx\; \tilde{e}(L_{\mathbf{H}};\overline{W}_s,\overline{W}_n) \triangleq \int_t\int_f \Big[\big|1-L_{\mathbf{H}}(t,f)\big|^2 \cdot \overline{W}_s(t,f) + \big|L_{\mathbf{H}}(t,f)\big|^2\,\overline{W}_n(t,f)\Big]\,dt\,df\,,$$

$$L_{\mathbf{H}_W}(t,f) \;\approx\; L_{\widetilde{\mathbf{H}}_W}(t,f) \triangleq \frac{\overline{W}_s(t,f)}{\overline{W}_s(t,f)+\overline{W}_n(t,f)}\,, \tag{8}$$

$$e_{\min}(\mathbf{R}_s,\mathbf{R}_n) \;\approx\; \tilde{e}_{\min}(\overline{W}_s,\overline{W}_n) \triangleq \int_t\int_f \frac{\overline{W}_s(t,f)\overline{W}_n(t,f)}{\overline{W}_s(t,f)+\overline{W}_n(t,f)}\,dt\,df\,. \tag{9}$$

In analogy to (4), we define the *minimax robust TF Wiener filter* $\widetilde{\mathbf{H}}_R$ via its Weyl symbol as

$$L_{\widetilde{\mathbf{H}}_R}(t,f) \triangleq \arg\min_{L_{\mathbf{H}}} \max_{\substack{\overline{W}_s\in\widetilde{\mathcal{S}}\\ \overline{W}_n\in\widetilde{\mathcal{N}}}} \tilde{e}(L_{\mathbf{H}};\overline{W}_s,\overline{W}_n)\,,$$

where $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{N}}$ are uncertainty classes[4] for $\overline{W}_s(t,f)$ and $\overline{W}_n(t,f)$. Assuming $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{N}}$ to be convex and proceeding in analogy to Section 2 and the stationary case, we can show that $\widetilde{\mathbf{H}}_R$ equals the ordinary TF Wiener filter in (8),

$$L_{\widetilde{\mathbf{H}}_R}(t,f) \;=\; L_{\widetilde{\mathbf{H}}_W^L}(t,f) \;=\; \frac{\overline{W}_s^L(t,f)}{\overline{W}_s^L(t,f)+\overline{W}_n^L(t,f)}\,, \tag{10}$$

calculated for least favorable pseudo-WVS

$$\big(\overline{W}_s^L,\overline{W}_n^L\big) \;=\; \arg\max_{\substack{\overline{W}_s\in\widetilde{\mathcal{S}}\\ \overline{W}_n\in\widetilde{\mathcal{N}}}} \tilde{e}_{\min}(\overline{W}_s,\overline{W}_n)$$

with $\tilde{e}_{\min}(\overline{W}_s,\overline{W}_n)$ given by (9). This generalizes a similar result in the stationary case [4]. From $L_{\widetilde{\mathbf{H}}_R}(t,f)$, $\widetilde{\mathbf{H}}_R$ can be obtained by an inverse Weyl transform [10, 11].

Next, we propose three different definitions of TF uncertainty classes $\widetilde{\mathcal{S}}$, $\widetilde{\mathcal{N}}$ and we provide closed-form expressions for the respective robust TF Wiener filters $\widetilde{\mathbf{H}}_R$.

***p*-Point Model.** Let $\{\mathcal{R}_i\}_{i=1,2,\ldots,N}$ be a partition of the TF plane, i.e., $\bigcup_{i=1}^N \mathcal{R}_i = \mathbb{R}^2$ and $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ for $i \neq j$. Extending the stationary case definition in [3, 5], so-called *p-point uncertainty classes* can be defined for WVS as [8]

$$\widetilde{\mathcal{S}} = \Big\{\overline{W}_s(t,f):\ \iint_{\mathcal{R}_i} \overline{W}_s(t,f)\,dt\,df = s_i\,,\ \ i=1,2,\ldots,N\Big\}$$

$$\widetilde{\mathcal{N}} = \Big\{\overline{W}_n(t,f):\ \iint_{\mathcal{R}_i} \overline{W}_n(t,f)\,dt\,df = n_i\,,\ \ i=1,2,\ldots,N\Big\},$$

i.e., as the sets that contain all pseudo-WVS having prescribed energies $s_i \geq 0$ and $n_i \geq 0$ in prescribed TF regions $\mathcal{R}_i$. The sets $\widetilde{\mathcal{S}}$, $\widetilde{\mathcal{N}}$ are easily shown to be convex.

A TF reformulation of the results in [8, 3] yields as least favorable pseudo-WVS $\overline{W}_s^L(t,f) = \sum_{i=1}^N \overline{W}_{s,i}(t,f)$ and $\overline{W}_n^L(t,f) = \sum_{i=1}^N \overline{W}_{n,i}(t,f)$, where $\overline{W}_{s,i}(t,f)$ and $\overline{W}_{n,i}(t,f)$ are arbitrary nonnegative functions that are zero outside $\mathcal{R}_i$ and satisfy $n_i\overline{W}_{s,i}(t,f) = s_i\overline{W}_{n,i}(t,f)$. The robust TF Wiener filter in (10) is then obtained as

$$L_{\widetilde{\mathbf{H}}_R}(t,f) \;=\; \sum_{i=1}^N w_i\, I_{\mathcal{R}_i}(t,f) \quad \text{with } w_i = \frac{s_i}{s_i+n_i}\,, \tag{11}$$

where $I_{\mathcal{R}_i}(t,f)$ is the indicator function of $\mathcal{R}_i$. Note that $L_{\widetilde{\mathbf{H}}_R}(t,f)$ is piecewise constant, expressing constant TF weighting in a given TF region $\mathcal{R}_i$. Furthermore, $\widetilde{\mathbf{H}}_R$ can be shown to yield a constant TF MSE $\tilde{e}(L_{\widetilde{\mathbf{H}}_R};\overline{W}_s,\overline{W}_n) = \sum_{i=1}^N \frac{s_i n_i}{s_i+n_i}$ for all $\overline{W}_s \in \widetilde{\mathcal{S}}$, $\overline{W}_n \in \widetilde{\mathcal{N}}$.

It has been shown [8] that $\widetilde{\mathbf{H}}_R$ in (11) is a good approximation to the analogous robust time-varying Wiener filter $\mathbf{H}_R$ defined according to (4). Thus, our TF formulation of robust time-varying Wiener filters is valid, and (since $\mathbf{H}_R$ is not based on an underspread assumption) $\widetilde{\mathbf{H}}_R$ is robust also for processes that are not underspread.

An intuitive and computationally efficient approximate TF implementation of the robust TF filter $\widetilde{\mathbf{H}}_R$ in (11) exists if the partition $\{\mathcal{R}_i\}$ corresponds to a uniform rectangular tiling of the TF plane, i.e., the TF regions are chosen as $\mathcal{R}_{k,l} = [kT-T/2, kT+T/2) \times [lF-F/2, lF+F/2)$ with $TF = M \in \mathbb{N}$ (note that now we use a double index). Let $\{x^{(m)}(t)\}_{m=1,2,\ldots,M}$ denote an orthonormal basis for the signal subspace $\mathcal{X}_{0,0}$ corresponding to the TF rectangle $\mathcal{R}_{0,0}$ (this correspondence is defined in [17]). Since $\mathcal{R}_{k,l}$ is obtained from $\mathcal{R}_{0,0}$ through a TF shift by (kT, lF), an orthonormal basis for the signal subspace $\mathcal{X}_{k,l}$ corresponding to $\mathcal{R}_{k,l}$ is given by $\{x_{k,l}^{(m)}(t) = x^{(m)}(t-kT)\,e^{j2\pi lFt}\}_{m=1,2,\ldots,M}$ [17]. We now propose to approximate $\widetilde{\mathbf{H}}_R$ in (11) (to be more precise, $\mathbf{H}_R$) by the filter $\widehat{\mathbf{H}}_R \triangleq \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} w_{k,l}\,\mathbf{P}_{k,l}$ with $w_{k,l} = \frac{s_{k,l}}{s_{k,l}+n_{k,l}}$, where $\mathbf{P}_{k,l}$ is the orthogonal projection operator on $\mathcal{X}_{k,l}$. The resulting signal estimate can then be expressed as

$$(\widehat{\mathbf{H}}_R r)(t) \;=\; \sum_{m=1}^{M}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} w_{k,l}\, G_r^{(m)}(k,l)\, x_{k,l}^{(m)}(t)\,,$$

with the *Gabor coefficients* [18] $G_r^{(m)}(k,l) = \langle r, x_{k,l}^{(m)}\rangle = \int_{-\infty}^{\infty} r(t)\,x^{(m)*}(t-kT)\,e^{-j2\pi lFt}\,dt$, $m = 1,2,\ldots,M$. Thus, $\widehat{\mathbf{H}}_R$ is a *multi-window* [18] *Gabor filter* consisting of Gabor analysis, multiplicative modification, and Gabor synthesis in each of the M branches.

If the partition $\{\mathcal{R}_i\}$ is a wavelet-type tiling of the TF plane, a (conceptually analogous) multi-wavelet implementation of the robust Wiener filter can be developed.

Variational Neighborhood Model. Let $\overline{W}_s^0(t,f)$ and $\overline{W}_n^0(t,f)$ be nominal pseudo-WVS with mean energies $\bar{E}_s^0 = \int_t\int_f \overline{W}_s^0(t,f)\,dt\,df$ and $\bar{E}_n^0 = \int_t\int_f \overline{W}_n^0(t,f)\,dt\,df$. Extending the stationary case [4, 5], we define *variational neighborhood uncertainty classes* for WVS as

$$\widetilde{\mathcal{S}} \;=\; \Big\{\overline{W}_s(t,f):\ \big\|\overline{W}_s - \overline{W}_s^0\big\|_1 \leq \epsilon\bar{E}_s^0\Big\}$$

$$\widetilde{\mathcal{N}} \;=\; \Big\{\overline{W}_n(t,f):\ \big\|\overline{W}_n - \overline{W}_n^0\big\|_1 \leq \epsilon\bar{E}_n^0\Big\}\,,$$

with fixed $\epsilon > 0$, combined with the requirement of fixed mean energies $\int_t\int_f \overline{W}_s(t,f)\,dt\,df = \bar{E}_s^0$ and $\int_t\int_f \overline{W}_n(t,f)\,dt\,df = \bar{E}_n^0$. The sets $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{N}}$ can be shown to be convex.

[3]The tilde will indicate TF approximations or TF versions.

[4]Note that $\widetilde{\mathcal{S}}$, $\widetilde{\mathcal{N}}$ are TF analogues of $\mathcal{S}$, $\mathcal{N}$. Here and in what follows, $\overline{W}_s(t,f)$ and $\overline{W}_n(t,f)$ are "pseudo-WVS" that are not necessarily valid WVS but arbitrary TF functions that are (essentially) nonnegative. (We note that the WVS of an underspread process is essentially nonnegative [14, 15].)

In what follows, we define the *nominal TF SNR* $\mathrm{SNR}^0(t,f) \triangleq \overline{W}_s^0(t,f)/\overline{W}_n^0(t,f)$ and use the abbreviation $\overline{W}^0(t,f) \triangleq \bar{E}_n^0\overline{W}_s^0(t,f)+\bar{E}_s^0\overline{W}_n^0(t,f)$. Extending [4], it can be shown that the least favorable pseudo-WVS are given by

$$\overline{W}_s^L(t,f) = \begin{cases} \frac{c_1}{\bar{E}_s^0+c_1\bar{E}_n^0}\overline{W}^0(t,f) & \text{for } (t,f)\in\mathcal{R}_1 \\ \overline{W}_s^0(t,f) & \text{for } (t,f)\in\mathcal{R}_0 \\ \frac{c_2}{\bar{E}_s^0+c_2\bar{E}_n^0}\overline{W}^0(t,f) & \text{for } (t,f)\in\mathcal{R}_2\,, \end{cases}$$

$$\overline{W}_n^L(t,f) = \begin{cases} \frac{1}{\bar{E}_s^0+c_1\bar{E}_n^0}\overline{W}^0(t,f) & \text{for } (t,f)\in\mathcal{R}_1 \\ \overline{W}_n^0(t,f) & \text{for } (t,f)\in\mathcal{R}_0 \\ \frac{1}{\bar{E}_s^0+c_2\bar{E}_n^0}\overline{W}^0(t,f) & \text{for } (t,f)\in\mathcal{R}_2\,. \end{cases}$$

Here $\mathcal{R}_1$, $\mathcal{R}_0$, and $\mathcal{R}_2$ are the TF regions where $\mathrm{SNR}^0(t,f)$ is $<c_1$, $\in[c_1,c_2]$, and $>c_2$, respectively, and the constants c_1, c_2 are chosen such that $\|\overline{W}_s^L - \overline{W}_s^0\|_1 = \epsilon\bar{E}_s^0$ and $\|\overline{W}_n^L - \overline{W}_n^0\|_1 = \epsilon\bar{E}_n^0$ (which is always possible if $\mathcal{S}\cap\mathcal{N}=\emptyset$). The corresponding TF SNR, $\mathrm{SNR}^L(t,f) \triangleq \overline{W}_s^L(t,f)/\overline{W}_n^L(t,f)$, equals c_1, $\mathrm{SNR}^0(t,f)$, and c_2 on $\mathcal{R}_1$, $\mathcal{R}_0$, and $\mathcal{R}_2$, respectively, i.e., $\mathrm{SNR}^L(t,f)$ is $\mathrm{SNR}^0(t,f)$ clipped from below and above. The Weyl symbol of the robust TF Wiener filter in (10) is then obtained as

$$L_{\widetilde{\mathbf{H}}_R}(t,f) = \begin{cases} L_{\min} & \text{for } (t,f)\in\mathcal{R}_1 \\ L_{\widetilde{\mathbf{H}}_W^0}(t,f) & \text{for } (t,f)\in\mathcal{R}_0 \\ L_{\max} & \text{for } (t,f)\in\mathcal{R}_2\,, \end{cases} \tag{12}$$

with $L_{\widetilde{\mathbf{H}}_W^0}(t,f) = \overline{W}_s^0(t,f)/[\overline{W}_s^0(t,f)+\overline{W}_n^0(t,f)]$ and $L_{\min} = \frac{c_1}{1+c_1}$, $L_{\max} = \frac{c_2}{1+c_2}$. Thus, $L_{\widetilde{\mathbf{H}}_R}(t,f)$ is a clipped version of the Weyl symbol of the nominal TF Wiener filter, $L_{\widetilde{\mathbf{H}}_W^0}(t,f)$. Indeed, the potential performance loss of $\widetilde{\mathbf{H}}_W^0$ is due to $L_{\widetilde{\mathbf{H}}_W^0}(t,f)$ being too close to 0 (to 1) in TF regions where $\mathrm{SNR}^0(t,f)$ is very small (large), resulting in a filter attenuation (gain) that is too strong for *non*-nominal WVS. Hence, a clipping of $L_{\widetilde{\mathbf{H}}_W^0}(t,f)$ (which implies the clipping $\mathrm{SNR}^0(t,f)\to\mathrm{SNR}^L(t,f)$ since $L_{\widetilde{\mathbf{H}}_W^0}(t,f) = \mathrm{SNR}^0(t,f)/[\mathrm{SNR}^0(t,f)+1]$) results in robustness.

ϵ-Contamination Model. Again extending the stationary case [2], we define *ϵ-contamination uncertainty classes*

$$\widetilde{\mathcal{S}} = \Big\{\overline{W}_s(t,f):\ \overline{W}_s(t,f) = (1-\epsilon)\overline{W}_s^0(t,f) + \epsilon\overline{W}_s'(t,f)\Big\}$$

$$\widetilde{\mathcal{N}} = \Big\{\overline{W}_n(t,f):\ \overline{W}_n(t,f) = (1-\epsilon)\overline{W}_n^0(t,f) + \epsilon\overline{W}_n'(t,f)\Big\},$$

with fixed $\epsilon>0$, where $\overline{W}_s'(t,f)\geq 0$, $\overline{W}_n'(t,f)\geq 0$ are arbitrary up to the usual constraint of fixed mean energy, i.e., $\int_t\int_f \overline{W}_s'(t,f)\,dt\,df = \bar{E}_s^0$ and $\int_t\int_f \overline{W}_n'(t,f)\,dt\,df = \bar{E}_n^0$. The sets $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{N}}$ can be shown to be convex.

The least favorable pseudo-WVS are here obtained as

$$\overline{W}_s^L(t,f) = \begin{cases} c_1(1-\epsilon)\overline{W}_n^0(t,f) & \text{for } (t,f)\in\mathcal{R}_1\,, \\ (1-\epsilon)\overline{W}_s^0(t,f) & \text{for } (t,f)\in\mathcal{R}_0\cup\mathcal{R}_2\,, \end{cases}$$

$$\overline{W}_n^L(t,f) = \begin{cases} \frac{1}{c_2}(1-\epsilon)\overline{W}_s^0(t,f) & \text{for } (t,f)\in\mathcal{R}_2\,, \\ (1-\epsilon)\overline{W}_n^0(t,f) & \text{for } (t,f)\in\mathcal{R}_0\cup\mathcal{R}_1\,, \end{cases}$$

with c_1, c_2 chosen such that $\overline{W}_s^L(t,f)$, $\overline{W}_n^L(t,f)$ meet the mean energy constraints. The corresponding TF SNR is again a clipped version of $\mathrm{SNR}^0(t,f)$, i.e., $\mathrm{SNR}^L(t,f)$ equals c_1, $\mathrm{SNR}^0(t,f)$, and c_2 on $\mathcal{R}_1$, $\mathcal{R}_0$, and $\mathcal{R}_2$, respectively. Furthermore, the Weyl symbol of the robust TF Wiener filter in (10) equals the clipped version of $L_{\widetilde{\mathbf{H}}_W^0}(t,f)$ given in (12). Note, however, that $\mathcal{R}_1$, $\mathcal{R}_0$, $\mathcal{R}_2$ and $L_{\min}$, $L_{\max}$ are different due to the different uncertainty model.

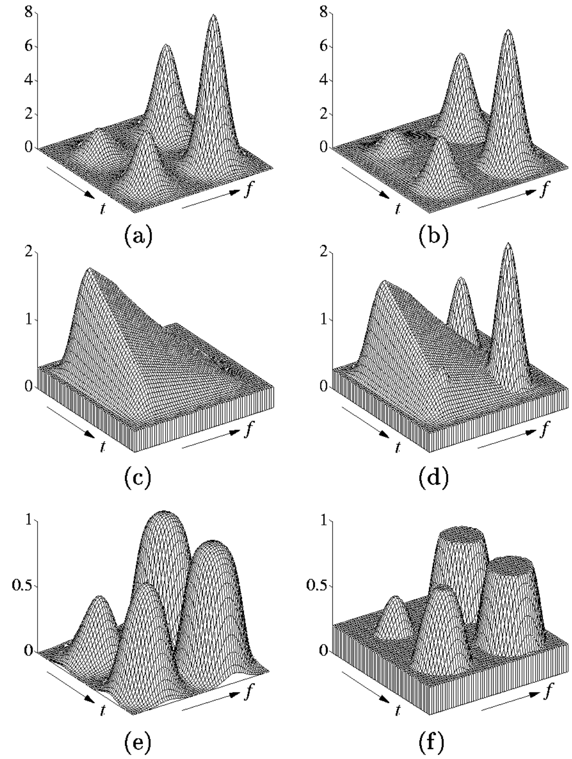

Figure 1. *TF representations of signal and noise statistics as well as nominal and robust TF Wiener filters for ϵ-contamination model ($\epsilon = 0.1$): (a) $\overline{W}_s^0(t,f)$, (b) $\overline{W}_s^L(t,f)$, (c) $\overline{W}_n^0(t,f)$, (d) $\overline{W}_n^L(t,f)$, (e) $L_{\widetilde{\mathbf{H}}_W^0}(t,f)$, (f) $L_{\widetilde{\mathbf{H}}_R}(t,f)$.*

4 SIMULATION RESULTS

Figs. 1(a) and 1(c) show nominal WVS of signal and noise. The least favorable WVS obtained for an ϵ-contamination model with $\epsilon = 0.1$ are depicted in Figs. 1(b) and 1(d). Fig. 1(f) shows that the Weyl symbol of the minimax robust TF Wiener filter $\widetilde{\mathbf{H}}_R$ is indeed a clipped version (with $L_{\min} = 0.21$, $L_{\max} = 0.77$) of the Weyl symbol of the nominal TF Wiener filter $\widetilde{\mathbf{H}}_W^0$ depicted in Fig. 1(e).

Table 1 compares the MSEs achieved by $\widetilde{\mathbf{H}}_W^0$ and $\widetilde{\mathbf{H}}_R$ at nominal operating conditions $(\overline{W}_s^0, \overline{W}_n^0)$ and at least favorable operating conditions $(\overline{W}_s^L, \overline{W}_n^L)$ for several values of ϵ. It is seen that the MSE variation is much smaller for $\widetilde{\mathbf{H}}_R$ than for[5] $\widetilde{\mathbf{H}}_W^0$, i.e., $\widetilde{\mathbf{H}}_R$ is indeed robust with respect to a variation of operating conditions. We note that simulation results for the p-point model can be found in [8].

5 CONCLUSION

We have introduced minimax robust time-varying Wiener filters that guarantee a certain performance within given

[5]Here, it should be noted that while for $\widetilde{\mathbf{H}}_R$ the worst-case operating conditions are given by $(\overline{W}_s^L, \overline{W}_n^L)$, the performance of $\widetilde{\mathbf{H}}_W^0$ can be worse than at $(\overline{W}_s^L, \overline{W}_n^L)$.

ϵ	0.01	0.05	0.10	0.20	0.40
$\tilde{e}(L_{\widetilde{\mathbf{H}}_W^0};\overline{W}_s^0,\overline{W}_n^0)$	9.65	9.65	9.65	9.65	9.65
$\tilde{e}(L_{\widetilde{\mathbf{H}}_W^0};\overline{W}_s^L,\overline{W}_n^L)$	10.35	12.60	15.66	20.99	30.64
$\tilde{e}(L_{\widetilde{\mathbf{H}}_R};\overline{W}_s^0,\overline{W}_n^0)$	9.69	9.99	10.74	12.90	19.53
$\tilde{e}(L_{\widetilde{\mathbf{H}}_R};\overline{W}_s^L,\overline{W}_n^L)$	10.33	12.26	14.48	17.40	19.55

Table 1. *MSE obtained with $\widetilde{\mathbf{H}}_W^0$ and $\widetilde{\mathbf{H}}_R$ at nominal operating conditions $(\overline{W}_s^0,\overline{W}_n^0)$ and at least favorable operating conditions $(\overline{W}_s^L,\overline{W}_n^L)$ for several values of ϵ.*

uncertainty classes of nonstationary processes. A time-frequency reformulation of the minimax theory allowed us to replace the calculus of operators by the simpler calculus of functions. Intuitively appealing and simple closed-form expressions of robust time-frequency Wiener filters have been obtained for three important uncertainty models.

APPENDIX: PROOF OF THEOREM 2.1

We show that (6) is necessary and sufficient for $\mathbf{R}_s^L$, $\mathbf{R}_n^L$ to satisfy the left-hand inequality in (7) with $\mathbf{H}_L=\mathbf{H}_W^L$,

$$e(\mathbf{H}_W^L;\mathbf{R}_s,\mathbf{R}_n)\;\le\;e(\mathbf{H}_W^L;\mathbf{R}_s^L,\mathbf{R}_n^L)\,. \tag{13}$$

Our proof (see [19] for more details) is essentially an adaptation and combination of arguments in [4, 7].

To show that (6) is necessary for (13), we combine (13) with $e_{\min}(\mathbf{R}_s,\mathbf{R}_n)\le e(\mathbf{H}_W^L;\mathbf{R}_s,\mathbf{R}_n)$ and $e(\mathbf{H}_W^L;\mathbf{R}_s^L,\mathbf{R}_n^L)=e_{\min}(\mathbf{R}_s^L,\mathbf{R}_n^L)$ to obtain $e_{\min}(\mathbf{R}_s,\mathbf{R}_n)\le e_{\min}(\mathbf{R}_s^L,\mathbf{R}_n^L)$ for all $\mathbf{R}_s\in\mathcal{S}$, $\mathbf{R}_n\in\mathcal{N}$, which is (6).

We now prove that (6) is sufficient for (13). Let $\mathbf{R}_s\in\mathcal{S}$ and $\mathbf{R}_n\in\mathcal{N}$. One can show [19] that $e_{\min}(\mathbf{R}_s,\mathbf{R}_n)$ is a concave function of $\mathbf{R}_s$ and $\mathbf{R}_n$, so that

$$e_{\min}(\mathbf{R}_s^\alpha,\mathbf{R}_n^\alpha)\;\ge\;\alpha\,e_{\min}(\mathbf{R}_s,\mathbf{R}_n)+(1-\alpha)\,e_{\min}(\mathbf{R}_s^L,\mathbf{R}_n^L) \tag{14}$$

for $0\le\alpha\le 1$, where $\mathbf{R}_s^\alpha=\alpha\mathbf{R}_s+(1-\alpha)\mathbf{R}_s^L$, $\mathbf{R}_n^\alpha=\alpha\mathbf{R}_n+(1-\alpha)\mathbf{R}_n^L$. Due to the convexity of $\mathcal{S}$ and $\mathcal{N}$, we have $\mathbf{R}_s^\alpha\in\mathcal{S}$ and $\mathbf{R}_n^\alpha\in\mathcal{N}$ for $0\le\alpha\le 1$. Subtracting $e_{\min}(\mathbf{R}_s^L,\mathbf{R}_n^L)$ from both sides of (14) and dividing by α yields

$$0\;\ge\;\frac{1}{\alpha}f(\alpha)\;\ge\;e_{\min}(\mathbf{R}_s,\mathbf{R}_n)-e_{\min}(\mathbf{R}_s^L,\mathbf{R}_n^L)\,,$$

where $f(\alpha)\triangleq e_{\min}(\mathbf{R}_s^\alpha,\mathbf{R}_n^\alpha)-e_{\min}(\mathbf{R}_s^L,\mathbf{R}_n^L)$ and the upper bound follows from (6). Hence, $\frac{1}{\alpha}f(\alpha)$ is bounded, so that its limit for $\alpha\to 0^+$ exists and thus

$$0\;\ge\;\lim_{\alpha\to 0^+}\frac{1}{\alpha}f(\alpha)\,. \tag{15}$$

Let $\mathbf{R}_r=\mathbf{R}_s+\mathbf{R}_n$, $\mathbf{R}_r^\alpha=\mathbf{R}_s^\alpha+\mathbf{R}_n^\alpha$, and $\mathbf{R}_r^L=\mathbf{R}_s^L+\mathbf{R}_n^L$. Using $e_{\min}(\mathbf{R}_s,\mathbf{R}_n)=\mathrm{tr}\{\mathbf{R}_s\}-\mathrm{tr}\{\mathbf{R}_s(\mathbf{R}_r)^{-1}\mathbf{R}_s\}$ (cf. (3)) and $\mathrm{tr}\{\mathbf{R}_s^\alpha\}=\mathrm{tr}\{\mathbf{R}_s^L\}$, we obtain $f(\alpha)=\mathrm{tr}\{\mathbf{R}_s^L(\mathbf{R}_r^L)^{-1}\mathbf{R}_s^L\}-\mathrm{tr}\{\mathbf{R}_s^\alpha(\mathbf{R}_r^\alpha)^{-1}\mathbf{R}_s^\alpha\}$. Separating terms and using RKHS techniques similar to [7] yields [19]

$$\lim_{\alpha\to 0^+}\frac{1}{\alpha}f(\alpha)=\mathrm{tr}\{\mathbf{H}_W^L\mathbf{R}_r\mathbf{H}_W^{L+}\}-\mathrm{tr}\{\mathbf{H}_W^L\mathbf{R}_r^L\mathbf{H}_W^{L+}\}$$
$$+\,2\,\mathrm{tr}\{\mathbf{H}_W^L\mathbf{R}_s^L\}-2\,\mathrm{Re}\{\mathrm{tr}\{\mathbf{H}_W^L\mathbf{R}_s\}\}\,.$$

Adding $\mathrm{tr}\{\mathbf{R}_s\}$ and subtracting $\mathrm{tr}\{\mathbf{R}_s^L\}$ (which is allowed since $\mathrm{tr}\{\mathbf{R}_s^L\}=\mathrm{tr}\{\mathbf{R}_s\}$) and using $e(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n)=\mathrm{tr}\{\mathbf{R}_s\}-2\,\mathrm{Re}\{\mathrm{tr}\{\mathbf{H}\mathbf{R}_s\}\}+\mathrm{tr}\{\mathbf{H}\mathbf{R}_r\mathbf{H}^+\}$ (cf. (1)), we obtain

$$\lim_{\alpha\to 0^+}\frac{1}{\alpha}f(\alpha)=e(\mathbf{H}_W^L;\mathbf{R}_s,\mathbf{R}_n)-e(\mathbf{H}_W^L;\mathbf{R}_s^L,\mathbf{R}_n^L)\,.$$

With (15), this finally yields (13).

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