EFFECTS OF COLORED NOISE ON THE PERFORMANCE OF LINEAR EQUALIZERS

Deva K. Borah *Rodney A. Kennedy*

Telecommunications Engineering Group
Research School of Information Sciences
and Engineering
The Institute of Advanced Studies
Australian National University
Canberra ACT 0200, Australia

Inbar Fijalkow

ETIS/ENSEA,
95014 Cergy-Pontoise Cdx, France.

ABSTRACT

Performance of equalizers depends on the discrete time model of the input signal and noise. The use of higher sampling rate results in colored noise when the bandwidth of the noise-limiting prefilter is not sufficiently large. It is shown that the mean square error (MSE) performance of linear equalizers becomes sensitive to the decision delay when the input noise is colored, and by using the appropriate delay, a significant improvement in the MSE can be achieved. This is in contrast to the behaviour observed in the white noise case well known in the literature. This behaviour is explained in time domain by showing the contributions to the MSE from the columns of the channel convolution matrix and the noise eigenvectors. In the frequency domain, it is shown that the equalizer exploits the noise correlation to improve the MSE.

1. INTRODUCTION

Fractionally spaced equalizers have a number of advantages over symbol spaced equalizers including lower timing phase sensitivity, reduced noise enhancement, and availability of superior channel identification strategies due to cyclostationarity [1, 2, 3]. Most work on these equalizers begin with a discrete time model, where the transmitted samples are corrupted by additive white Gaussian noise samples. However, since fractionally spaced equalizers normally use a noise-limiting filter (NLF), the bandwidth of this filter affects the output noise and the noise samples are colored in general. If an equalization algorithm uses a high sampling rate (much higher than the Nyquist rate) and assumes a white noise model, then the bandwidth of the analog prefilter must also be higher than the bandwidth of the signal. This means allowing extra noise outside the signal bandwidth to enter into the receiver so that the noise samples are white. Alternatively, the noise power level can be kept low by using a prefilter that allows only the signal bandwidth to pass through. But in that case, higher than the Nyquist sampling rate produces colored noise.

Although the white noise model has been thoroughly investigated in the literature, the same cannot be said about colored noise. Colored noise affects the performance of equalization algorithms which assume a white noise model. Even if algorithms, such as the least mean squares (LMS) algorithm and Constant Modulus Algorithm (CMA), do not explicitly assume a white noise model, still their performance is affected by colored noise. Therefore, it is appropriate to study the effects of colored noise on the performance of linear equalizers.

It is known that in the white noise model, the MSE performance remains reasonably unaffected by the choice of the decision delay of the equalizer, except delays corresponding to the ends of the equalizer. We show in this paper that for colored noise case this is not true in general, and the mean square error (MSE) is highly sensitive to the decision delay of the equalizer so that even the best delay for the white noise case may be the worst delay choice for colored noise. By choosing the appropriate delay, significant gain in MSE can be achieved. Our work provides strong motivation for the design of algorithms to obtain the best delay, for example [5]. We explain this behaviour by considering the dependence of MSE on the column vectors of the channel convolution matrix and the noise eigenvectors. For certain decision delays, favourable equalizer taps are obtained so that these vectors contribute less to the MSE. The behaviour is also explained in the frequency domain by showing that the equalizer exploits the noise correlation and provides significantly smaller MSE.

2. SYSTEM MODEL

Consider the transmission of a sequence of M-ary symbols $\{a_n\}$, linearly modulating a transmit filter $p(t)$. This signal is distorted by a time-invariant channel $c(t)$, and corrupted by additive noise $n(t)$ so that the received signal is

$$y(t) = \sum_i a_i h(t - iT) + n(t) \quad (1)$$

where T denotes the symbol interval, and $h(t) = p(t) \otimes c(t)$ is the convolution of the transmit filter and the multipath channel impulse response. The impulse response is assumed to be nonzero only over $0 \leq t < LT$. This signal is passed through a received analog prefilter of impulse response, $g(t)$, and sampled at the rate $1/T_r$, where $T_r = T/r$, and r is an integer denoting the oversampling rate. For communication systems with excess bandwidth, that is with nonzero frequency components for $| f |> 1/2T$, a sampling rate $r > 1$ may be needed for the sampled observations to be sufficient statistics in order to detect $\{a_i\}$ optimally.

3. LINEAR EQUALIZER

Let us denote the discrete time samples at the output of the noise-limiting filter as $\{r_n\}$. If a linear equalizer of length rN is used, then the output symbol spaced samples of the equalizer are

$$\alpha_n = \mathbf{f}^H(\mathbf{Ha} + \mathbf{w}) \quad (2)$$

where $\mathbf{f}$ is the fractionally spaced equalizer tap vector, $\mathbf{H}$ is the convolution matrix of the overall channel impulse response, consisting of the transmit filter $p(t)$, the multipath channel $c(t)$ and the receive prefilter $g(t)$. The noise $\mathbf{w}$ is the fractionally spaced noise sample vector at the output of the filter $g(t)$, and need not be necessarily white. The linear equalizer $\mathbf{f}$ minimizes the MSE, $E[| \alpha_n - a_{n-d} |^2]$, so that the optimal solution for the equalizer taps is

$$\mathbf{f} = (\mathbf{HH}^H + \mathbf{R}_w)^{-1}\mathbf{h}_d \quad (3)$$

where $\mathbf{h}_d$ denotes the dth column of $\mathbf{H}$, and it corresponds to the set of achievable delays $[0, N + L - 2]$. d is referred to as the decision delay. The noise correlation matrix is $\mathbf{R}_w = E[\mathbf{ww}^H]$.

4. SAMPLING RATE AND THE NOISE MODEL

Depending on the frequency response of the NLF, the fractionally spaced noise samples $\{w_n\}$ may be white or colored. The correlation between noise samples, $r_w(t_1, t_2)$, is given by

$$\begin{aligned} r_w(t_1, t_2) &= E[w(t_1)w^*(t_2)] \\ &= E[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t_1 - \tau)g^*(t_2 - \tau') \\ &\quad n(t_1)n(t_2)d\tau d\tau'] \\ &= \frac{N_0}{2}\int_{-\infty}^{\infty} g(t_1 - \tau)g(t_2 - \tau)d\tau \end{aligned}$$

For $t_2 - t_1 = lT_r$,

$$r_w(t_1, t_2) = r_w(lT_r) = \frac{N_0}{2}\int_{-\infty}^{\infty} | G(f) |^2 e^{j2\pi f l T_r} df \quad (4)$$

If $g(t)$ is a root-raised cosine (RRC) filter, $| G(f) |^2$ is a raised cosine filter frequency response. Hence the RHS integral of (4) is the time-domain expression for the raised cosine filter [4].

Many equalization algorithms are based on explicit estimation of the channel impulse response. Many blind algorithms too estimate the channel impulse response and then perform equalization [2, 6]. If the equalizer taps are computed from the estimated channel impulse response, then three different cases can be considered.

Case 1: The algorithm assumes white noise model and the true noise samples are white. In that case the MSE becomes

$$\text{MSE} = 1 - \mathbf{h}_d^H(\mathbf{HH}^H + \sigma_w^2\mathbf{I})^{-1}\mathbf{h}_d.$$

The computation of the equalizer taps by this approach requires an estimate of the noise variance σ_w^2.

Case 2: The algorithm assumes colored noise model, and the true noise samples are colored. The MSE in this case is

$$\text{MSE} = 1 - \mathbf{h}_d^H(\mathbf{HH}^H + \mathbf{R}_w)^{-1}\mathbf{h}_d.$$

Note that the computation of the equalizer taps by this method requires an estimate of the noise correlation matrix $\mathbf{R}_w$.

Case 3: The algoritm assumes white noise model, but the true noise samples are colored. The MSE then becomes

$$\text{MSE} = E[| \mathbf{f}^H(\mathbf{Ha} + \mathbf{w}) - a_{n-d} |^2]$$

Using $\mathbf{f} = (\mathbf{HH}^H + \sigma_w^2\mathbf{I})^{-1}\mathbf{h}_d$, where σ_w^2 is the estimated noise variance assuming white noise model, the MSE expression can be simplified to

$$\begin{aligned} \text{MSE} &= 1 - \mathbf{h}_d^H(\mathbf{HH}^H + \sigma_w^2\mathbf{I})^{-1}\mathbf{h}_d + \mathbf{h}_d^H(\mathbf{HH}^H \\ &\quad + \sigma_w^2\mathbf{I})^{-1}(\mathbf{R}_w + \sigma_w^2\mathbf{I})(\mathbf{HH}^H + \sigma_w^2\mathbf{I})^{-1}\mathbf{h}_d \end{aligned}$$

The fourth case of assuming a colored noise model when the true noise samples are white is not important from a practical viewpoint.

4.1. Dependence on delay

The decision delay, d, plays an important role in the MSE of the equalizer. The received sample vector $\mathbf{y}$ may be expressed as

$$\mathbf{y} = \mathbf{Ha} + \mathbf{w} = \tilde{\mathbf{H}}\tilde{\mathbf{a}}$$

where $\tilde{\mathbf{H}} = [\mathbf{H}, \mathbf{v}_1, \cdots, \mathbf{v}_{rN}]$, and $\tilde{\mathbf{a}} = [\mathbf{a}, b_1, \cdots, b_{rN}]$. The vectors $\{\mathbf{v}_i\}$ are the eigenvectors of the correlation matrix $\mathbf{R}_w$, and $\{b_i\}$ are zero mean uncorrelated random variables (KL expansion). Then the MSE becomes

$$\text{MSE} = E[| \mathbf{f}^H \tilde{\mathbf{H}} \tilde{\mathbf{a}} - a_{n-d} |^2]$$

Assuming the elements of $\tilde{\mathbf{a}}$ to be uncorrelated, the MSE becomes

$$\text{MSE} = | 1 - \mathbf{f}^H \mathbf{h}_d |^2 + \sum_{i=0, i \neq d}^{K} | \mathbf{f}^H \mathbf{h}_i |^2 + \sum_{i=1}^{rN} \lambda_i | \mathbf{f}^H \mathbf{v}_i |^2 \quad (5)$$

where $K = L + N - 2$. The MSE consists of three terms: (i) the first term $| 1 - \mathbf{f}^H \mathbf{h}_d |^2$ represents the power of the offset of the equalized symbol from unity. (ii) the second term $\sum_{i=0,i\neq d}^{K} | \mathbf{f}^H \mathbf{h}_i |^2$ is the contribution to the MSE due to residual ISI, and (iii) the third term $\sum_{i=1}^{rN} \lambda_i | \mathbf{f}^H \mathbf{v}_i |^2$ contains contribution from noise. Since each term is positive, we want each to be minimum for the MSE to be minimum.

Assuming the matrix $\mathbf{H}\mathbf{H}^H + \mathbf{R}_w$ to be positive definite, the Cholesky decomposition provides $\mathbf{H}\mathbf{H}^H + \mathbf{R}_w = \mathbf{L}\mathbf{L}^H$, so that the contribution to the MSE due to the ith column $(i \neq d)$ of $\mathbf{H}$ is

$$| \mathbf{f}^H \mathbf{h}_i |^2 = | (\Gamma^H \mathbf{h}_d)^H (\Gamma^H \mathbf{h}_i) |^2 \quad (6)$$

where $\Gamma = \mathbf{L}^{-1}$. Similarly the contribution due to the m th noise eigenvector is

$$\lambda_m | \mathbf{f}^H \mathbf{h}_{K+m} |^2 = \lambda_m | (\Gamma^H \mathbf{h}_d)^H (\Gamma^H \mathbf{h}_i) |^2 \quad (7)$$

Equation (5) shows that the contribution to the MSE comes from all the columns of $\tilde{\mathbf{H}}$. Equations (6) and (7) show that the contribution to the MSE from the ith column of $\tilde{\mathbf{H}}$ matrix comes through an inner product with column $\mathbf{h}_d$ (associated with delay d) in a transformed space. For colored noise, the noise eigenvalues are not equal along the eigenvectors. Therefore, for some values of d, the inner product of the dominant eigenvectors with $\mathbf{h}_d$ in the transformed space may contribute large values to the MSE. For some other values of d, the MSE may be very small.

5. NUMERICAL RESULTS AND DISCUSSION

The MSE performance of linear equalizers in the presence of white and colored noise is studied. The trasmit filter $p(t)$ is an RRC filter $z_\alpha(t)$ with roll-off factor $\alpha = 0.35$. It is truncated to 8 symbol periods. The multipath channel $c(t)$ is

$$c(t) = z_\alpha(t) - 0.7 z_\alpha(t - \frac{11T}{12}) \quad (8)$$

The signal-to-noise ratio (SNR), i.e., the bit energy to normalized noise spectral density is E_b/N_0, where $E_b = \sum_l | h(lT_r) |^2$, and N_0 is the noise power spectral density at the input to the prefilter. The receive filter $g(t)$ is also an RRC filter with $\alpha = 0.35$. Note that if the transmitted signal is bandlimited to $| f | \leq (1-\alpha)/T$, the use of a receive RRC filter bandlimited to the same frequency results in colored noise in general. To obtain white noise samples, the receive RRC filter should have a roll-off over $r/2T$, thus, passing excess noise into the receiver. The noise samples are assumed to have passed through a perfect RRC filter, so that the noise correlation matrix can be directly obtained from (4). The equalizer consists of 24 fractionally spaced taps. Figure 1 shows the MSE versus delay d at an SNR of 15 dB. For white noise, the MSE is not very sensitive in the middle region of the allowed delays. However, for colored noise, the MSE becomes very sensitive to delay d. At sampling rate $r = 4$, all delays except 5 and 6 produce significantly smaller MSE. This behaviour is peculiar from the white noise case where delays 5 and 6 provide very good MSE. This behaviour can be explained by reference to Fig.2, where contributions to the MSE from columns $\mathbf{h}_i$, and noise eigenvectors $\mathbf{v}_i$ are shown. The horizontal axis shows the vector number and the vertical axis shows the contribution calculated from (5), (6), (7). For this numerical example, the first 12 vectors represent $\mathbf{h}_i$, and the remaining 24 vectors correspond to the noise eigenvectors with the corresponding eigenvalues in decreasing order. The figure shows that for delays 4 and 7 (small MSE), the contribution to the MSE due to the noise eigenvectors is very small, whereas for delays 5 and 6, the contribution is high.

Figure 3 shows the frequency domain behaviour of the channel and the equalizer in white and colored noise. Since due to the low pass NLF, there is no noise outside the bandwidth of the NLF, the equalizer has flexibility in its response in this frequency range. Since any response in this range is possible without noise enhancement[1]. The equalizer taps in the white noise case does not have this flexibility, since a large response at any frequency will also enhance noise. Finally, whereas we do see an improvement for the suboptimal linear equalizer, the performance improvement with the sampling rate is irrelevent for an optimal detector.

6. CONCLUSION

Because of the presence of the noise-limiting filter, the noise samples present at the input to a fractionally spaced equalizer are colored at higher sampling rates. This paper shows that by exploiting this noise correlation, the MSE performance of a linear equalizer can be significantly improved. The performance of the equalizer depends strongly on the

[1]In practice, the equalizer taps may be designed by considering the presence of a small amount of noise.

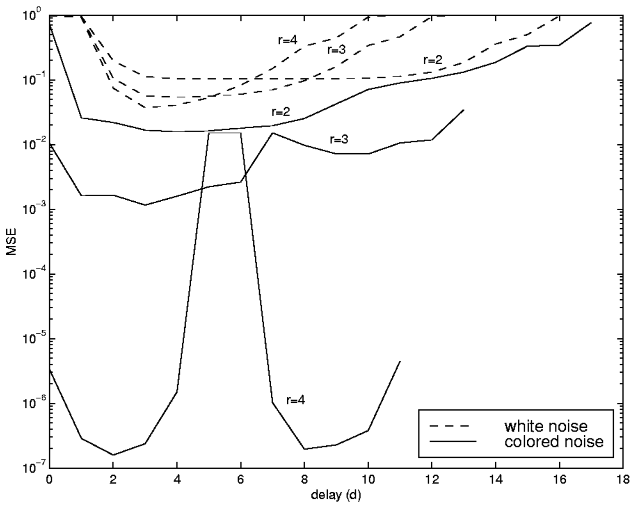

Figure 1: MSE at different delays

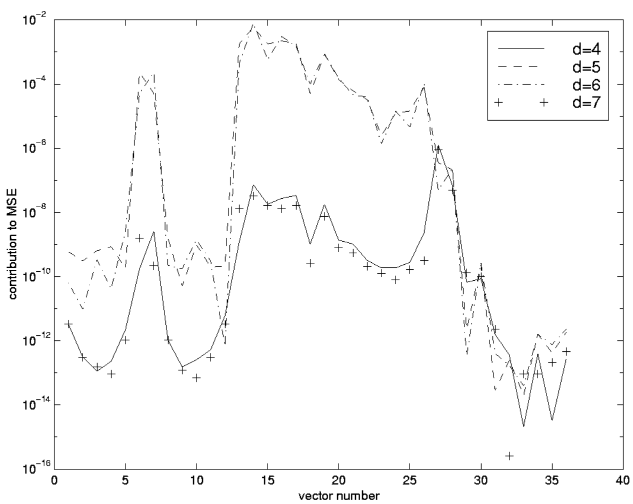

Figure 2: The effect on MSE due to the columns of the channel convolution matrix and noise eigenvectors

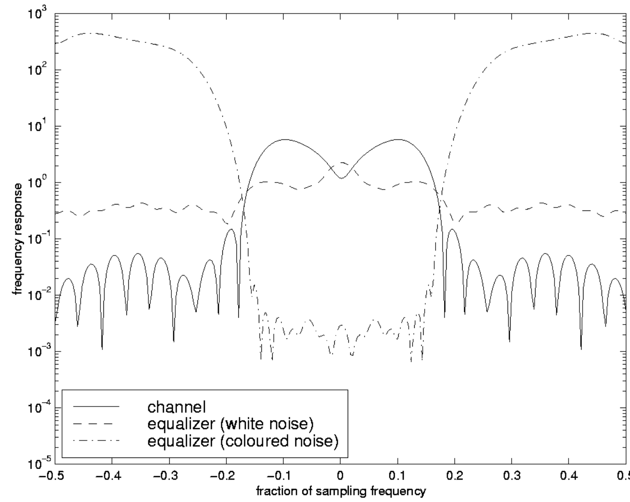

Figure 3: Frequency response of the channel and the equalizer

decision delay of detection, and the best delay for the white noise model may be the worst delay choice in colored noise. An explanation to this behaviour is presented in the time domain through the relationship between the MSE and the columns of the channel convolution matrix and the noise eigenvectors. The frequency response shows that in the colored noise case, the equalizer exploits the absence of noise outside the signal bandwidth. The equalizer in a white noise case does not have this flexibility.

7. REFERENCES

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