

HARMONIC RETRIEVAL IN NONSTATIONARY NOISE

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ABSTRACT

Harmonic retrieval is a classical signal processing problem but it has been almost invariably assumed that the additive noise is stationary. In this paper, we abandon this requirement and allow the additive noise to be nonstationary (but also non-cyclostationary in order to distinguish it from the information bearing signal). We show that various FFT based approaches can still be used on a single record of data to yield frequency estimates that have the $O(T^{-3})$ variance, where T is the data length. Stationary multiplicative noise is also permitted in the model. Numerical examples illustrate the key concepts of the paper.

1. INTRODUCTION

Harmonic retrieval has been a very important and active area of research in modern signal processing [2]. If all quantities involved are real-valued, a single component harmonic-in-noise process can be written as

$$x(t) = s(t) \cos(\omega_0 t + \phi_0) + v(t), \quad (1)$$

where $t = 0, 1, \dots, T-1$ is the discrete-time index. The objective is to estimate the frequency of interest ω_0 from the above single record of data.

The cases where (i) the amplitude $s(t)$ is a constant; i.e., $s(t) \equiv A$, and (ii) the amplitude $s(t)$ is a deterministic but time-varying function; e.g., $s(t) = A e^{-\lambda t}$, have been thoroughly investigated (see e.g., Walker [6], Hasan [3]). Recently, there has been greater interest towards studying (iii) the multiplicative noise case where the amplitude $s(t)$ is a stationary random process (see e.g., Swami [5], Zhou and Giannakis [7], [8]). In all cases (i)-(iii), the additive noise $v(t)$ is assumed to be stationary and the ω_0 estimate obtained using FFT based approaches has variance that is in the order of $O(T^{-3})$. In this paper, we focus on the case where $s(t)$ is a stationary random process (which includes $s(t) \equiv A$ as a special case) but abandon the stationarity assumption on $v(t)$. We will show that as long as $v(t)$ is not cyclostationary, FFT based algorithms can be formulated which yield frequency estimates that still maintain the $O(T^{-3})$ variance. These results are significant because they can explain why previous FFT based harmonic retrieval algorithms have been so successful in real life applications—they were devised using the stationary additive noise assumption but they also happened to work for nonstationary (excluding cyclostationary) $v(t)$ processes as well. There are also indirect applications of these results, an example of which is given here for the polynomial phase signals.

The paper is organized as follows: we derive FFT-based frequency estimation algorithms in Section 2 and show their variance expressions in Section 3. In Section 4, we illustrate the usefulness of our results by way of a polynomial phase

signal example where the optimum choice for the lag parameter is determined by applying the variance expression obtained in Section 3. Extensive computer simulation results are presented in Section 5 to illustrate the algorithms and the accuracy of the variance expressions. Finally, conclusions are drawn in Section 6.

2. FREQUENCY ESTIMATION

It is well known that an estimator based on the k th-order statistic of a random process has variance expression that depends on the 1st- through $(2k)$ th-order statistics of the same random process. Therefore higher-order statistics are often a necessity when deriving closed form variance expressions. Since the process in (1) is nonstationary, time-varying higher-order statistics need to be introduced.

2.1. Notations

For a given set of lags $\tau \triangleq (\tau_1, \dots, \tau_{k-1})$, the k th-order time-varying moment of $x(t)$ is defined as

$$m_{kx}(t; \tau) \triangleq E\{x(t)x(t+\tau_1) \dots x(t+\tau_{k-1})\}. \quad (2)$$

The corresponding time-varying moment spectrum is

$$M_{kx}(t; \omega) \triangleq \sum_{\tau} m_{kx}(t; \tau) e^{-j\tau\omega'}, \quad (3)$$

where $\omega \triangleq (\omega_1, \dots, \omega_{k-1})$.

It turns out that time-averaged statistics are useful when dealing with single record parameter estimation of nonstationary random processes. Similar to Zhou and Giannakis [7], we introduce the k th-order time-averaged moment:

$$\bar{m}_{kx}(\tau) \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} m_{kx}(t; \tau). \quad (4)$$

The k th-order time-averaged moment spectrum can be defined via $M_{kx}(t; \omega)$ or $\bar{m}_{kx}(\tau)$:

$$\bar{M}_{kx}(\omega) \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} M_{kx}(t; \omega), \quad (5)$$

$$= \sum_{\tau} \bar{m}_{kx}(\tau) e^{-j\tau\omega'}. \quad (6)$$

The k th-order time-varying cumulant of $x(t)$, $c_{kx}(t; \tau)$, can be obtained from $m_{lx}(t; \tau)$ with $1 \leq l \leq k$. For definitions and properties of cumulants, see Brillinger [1, p. 19]. k th-order time-varying cumulant spectrum $\bar{S}_{kx}(t; \omega)$, time-averaged cumulant $\bar{c}_{kx}(\tau)$, and time-averaged cumulant spectrum $\bar{S}_{kx}(\omega)$, can be defined similar to (3)-(6) with

the cumulant quantities replacing their moment counterparts. Note that for stationary processes, time-averaging has no effect; e.g., $\bar{c}_{ks}(\tau) = c_{ks}(\tau)$ for $s(t)$ stationary.

Next, we keep τ fixed, take the Fourier Series (FS) expansion of $c_{kx}(t; \tau)$ w.r.t. t , and define the k th-order cyclic moment of $x(t)$ as

$$C_{kx}(\alpha; \tau) \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} m_{kx}(t; \tau) e^{-j\alpha t}. \quad (7)$$

If $C_{kx}(\alpha; \tau) \neq 0$ for some $\alpha \neq 0$, then $x(t)$ is called k th-order cyclostationary. Conversely, if $C_{kx}(\alpha; \tau) = 0 \forall \alpha \neq 0$, then we say that $x(t)$ is not k th-order cyclostationary.

2.2. Assumptions

The following assumptions are made for the process in (1):

- (A1) ω_0 is a deterministic constant in $(0, \pi)$.
- (A2) ϕ_0 is a deterministic constant.
- (A3) $s(t)$ and $v(t)$ are mutually independent.
- (A4) $s(t)$ is stationary and mixing up to the 4th-order; i.e., the k th-order cumulant of $s(t)$ satisfies $\sum_{\tau} |c_{ks}(\tau)| < \infty$ for $1 \leq k \leq 4$.
- (A5) $v(t)$ is mixing up to the 4th-order; i.e., $\sum_{\tau} |c_{kv}(t; \tau)| < \infty$ for $1 \leq k \leq 4$ and $\forall t$.
- (A6) $v(t)$ is not 1st-order cyclostationary when $E[s(t)] \neq 0$ and $v(t)$ is not 2nd-order cyclostationary when $E[s(t)] = 0$. We emphasize that we do not make any assumption on the color or distribution of $s(t)$ and $v(t)$.

Because of (A4), the k th-order cumulant spectrum of $s(t)$ is finite; i.e., $|S_{ks}(\omega)| < \infty$. Similarly, the k th-order time-averaged cumulant spectrum of $v(t)$ is bounded, $|\bar{S}_{kv}(\omega)| < \infty$, due to (A5).

2.3. The nonzero mean case

Let us first consider the case where $m_s = E[s(t)] \neq 0$. The time-varying mean of $x(t)$ is

$$m_{1x}(t) = E[x(t)] = m_s \cos(\omega_0 t + \phi_0) + m_{1v}(t). \quad (8)$$

Taking its FS expansion w.r.t. t , we obtain,

$$C_{1x}(\alpha) = \frac{1}{2} m_s e^{j\phi_0} \delta_K(\alpha - \omega_0) + \frac{1}{2} m_s e^{-j\phi_0} \delta_K(\alpha + \omega_0) + \underbrace{\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} m_{1v}(t) e^{-j\alpha t}}_{C_{1v}(\alpha)}. \quad (9)$$

In the above equation, $\delta_K(\cdot)$ is the Kronecker delta function and it appears because

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} e^{j\alpha t} = \begin{cases} 1, & \alpha = 0 \text{ mod } (2\pi) \\ 0, & \text{otherwise} \end{cases} \triangleq \delta_K(\alpha). \quad (10)$$

Since we assume that $v(t)$ is not 1st-order cyclostationary, we have that $C_{1v}(\alpha) = 0, \forall \alpha \neq 0 \text{ mod } (2\pi)$. Therefore, (9) reduces to

$$C_{1x}(\alpha) = \frac{1}{2} m_s e^{j\phi_0} \delta_K(\alpha - \omega_0) + \bar{m}_{1v} \delta_K(\alpha), \quad (11)$$

where $\bar{m}_{1v} = \lim_{T \rightarrow \infty} T^{-1} \sum_{t=0}^{T-1} m_{1v}(t) = C_{1v}(0)$. Thus by searching $|C_{1x}(\alpha)|$ over $0 < \alpha < \pi$, we can locate a peak at $\alpha = \omega_0$ which yields the desired frequency information.

The same result can be derived from the 2nd-order time-averaged spectrum of $x(t)$ as well. We start with the time-varying auto-correlation of $x(t)$,

$$m_{2x}(t; \tau) \triangleq E[x(t)x(t+\tau)] = m_{2s}(\tau) \cos(\omega_0 t + \phi_0) \cos(\omega_0 t + \omega_0 \tau + \phi_0) + m_{2v}(t; \tau) \quad (12)$$

$$+ m_s m_{1v}(t + \tau) \cos(\omega_0 t + \phi_0) \quad (13)$$

$$+ m_s m_{1v}(t) \cos(\omega_0 t + \omega_0 \tau + \phi_0). \quad (15)$$

Using property (10), it follows easily that the asymptotic time average of (12) is $\frac{1}{2} m_{2s}(\tau) \cos(\omega_0 \tau)$. By definition, the asymptotic time average of (13) is $\bar{m}_{2v}(\tau)$. Our assumption (A4) with $k = 1$ implies that m_s is finite. This together with assumption (A6) ensure that the asymptotic time averages of (14) and (15) both tend to zero. Summarizing, we find

$$\bar{m}_{2x}(\tau) = \frac{1}{2} m_{2s}(\tau) \cos(\omega_0 \tau) + \bar{m}_{2v}(\tau). \quad (16)$$

The first term on the r.h.s. of (16) contributes the following to $\bar{M}_{2x}(\omega)$:

$$\frac{1}{2} S_{2s}(\omega_0) + \frac{m_s^2}{4} \delta_D(\omega \pm \omega_0), \quad (17)$$

where $\delta_D(\cdot)$ is the Dirac delta function. Next, we examine

$$\bar{m}_{2v}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \underbrace{m_{1v}(t) m_{1v}(t + \tau)}_{g_v(\tau)} + \bar{c}_{2v}(\tau).$$

According to (A6), $m_{1v}(t)$ is not periodic in t and hence $m_{1v}(t + \tau)$ is not periodic in τ . Therefore $g_v(\tau)$ defined above is not a periodic function of τ and its FT $G_v(\omega)$ is finite except possibly at $\omega = 0$. Recall also $|S_{2v}(\omega)| < \infty$ as a consequence of (A5). We therefore conclude that $|\bar{M}_{2v}(\omega)| < \infty$ except possibly at $\omega = 0$. This result, together with (17), implies that if we search over $\omega \in (0, \pi)$, we will find that $\bar{M}_{2x}(\omega)$ peaks only at $\omega = \omega_0$ which is the desired frequency.

From the above analyses, we realize that the same conclusion can be obtained whether we use cyclic or time-averaged spectral statistics. In the cyclic mean approach, a Kronecker delta appears at $\alpha = \omega_0$ and $|C_{1x}(\alpha)|$ is *theoretically* insensitive to the additive noise effect. Using the time-averaged spectrum concept, we find that a Dirac delta appears at $\omega = \omega_0$. Although the contribution of the additive noise term to $\bar{M}_{2x}(\omega)$ is nonzero, it nevertheless has finite amplitude at $\omega \neq 0$ which is basically “nothing” as compared to the Dirac delta at $\omega = \omega_0$.

A natural estimator for (9) is

$$\hat{C}_{1x}(\alpha) = \frac{1}{T} \sum_{t=0}^{T-1} x(t) e^{-j\alpha t}, \quad (18)$$

and the natural estimator for $\bar{M}_{2x}(\omega)$ is

$$\begin{aligned} \hat{\bar{M}}_{2x}(\omega) &= \sum_{\tau} \frac{1}{T} \sum_{t=0}^{T-1} x(t)x(t+\tau) e^{-j\omega\tau} \\ &= \frac{1}{T} \left| \sum_{t=0}^{T-1} x(t) e^{-j\omega t} \right|^2. \end{aligned} \quad (19)$$

Both (18) and (19) require taking the FT of the data. Their apparent relationship, $\hat{M}_{2x}(\omega) = T|\hat{C}_{1x}(\omega)|^2$, implies that peak picking either one yields the same $\hat{\omega}_0$ estimate. For simplicity, we adopt

$$\hat{\omega}_0 = \arg \max_{0 < \alpha < \pi} |\hat{C}_{1x}(\alpha)|. \quad (20)$$

2.4. The zero mean case

When $m_s = 0$, the time-varying correlation of $x(t)$ at lag $\tau = 0$ is given by

$$m_{2x}(t; 0) = E[x^2(t)] = \sigma_s^2 \cos^2(\omega_0 t + \phi_0) + m_{2v}(t; 0), \quad (21)$$

where $\sigma_s^2 = E[s^2(t)]$ is the variance of $s(t)$. Taking the FS expansion of (21) w.r.t. t and using (10), we obtain the cyclic correlation of $x(t)$ at cycle α and lag 0:

$$\begin{aligned} C_{2x}(\alpha; 0) &= \frac{1}{4} \sigma_s^2 e^{\mp 2j\phi_0} \delta_K(\alpha \pm 2\omega_0) \\ &+ \frac{1}{2} \sigma_s^2 \delta_K(\alpha) + C_{2v}(\alpha; 0). \end{aligned} \quad (22)$$

According to (A6), $C_{2v}(\alpha; 0) = 0, \forall \alpha \neq 0$. Therefore, by searching for peaks of $|C_{2x}(\alpha; 0)|$ over the range $0 < \alpha < \pi$, we can estimate $2\omega_0$, which in turn, yields an unambiguous estimate of ω_0 if it lies in $(0, \pi/2)$. When $\omega_0 > \pi/2$, wrap around occurs in $C_{2x}(\alpha; 0)$ and ω_0 cannot be distinguished from $\pi - \omega_0$.

Sample estimate of (22) is obtained as

$$\hat{C}_{2x}(\alpha; 0) = \frac{1}{T} \sum_{t=0}^{T-1} x^2(t) e^{-j\alpha t}, \quad (23)$$

which is nothing but the normalized FT of the data squared. Subsequently, the frequency estimate is obtained as

$$\hat{\omega}_0 = \frac{1}{2} \arg \max_{0 < \alpha < \pi} |\hat{C}_{2x}(\alpha; 0)|. \quad (24)$$

3. VARIANCE EXPRESSIONS

Following the procedures of Zhou and Giannakis [8], we can derive variance expressions for the sample estimators in (20) and (24). Most of the steps are similar although minor modifications need to be made to accommodate the fact that $c_{kv}(t; \tau)$ is present instead of $c_{kv}(\tau)$. Nevertheless, the final variance expressions look almost identical, except that the spectral quantities of $v(t)$ must now be replaced by their time-averaged counterparts.

For the nonzero mean case and the frequency estimator in (20), we find the following large sample (i.e., large T) variance expression:

$$\text{var}(\hat{\omega}_0) \approx \frac{1}{T^3} \left[\frac{24\bar{S}_{2v}(\omega_0)}{m_{s0}^2} + \frac{6S_{2s}(2\omega_0)}{m_{s0}^2} \right]. \quad (25)$$

We wish to point out that the above expression applies when $\omega_0 \neq \pi/2$ and when $2\omega_0$ is not a cycle of $c_{2v}(t; \tau)$; otherwise additional terms need to be added to (25).

For the zero mean case and the frequency estimator in (24), the variance expression is

$$\text{var}(\hat{\omega}_0) \approx \frac{1}{T^3} \frac{48G(\omega_0)}{\sigma_{s0}^4}, \quad (26)$$

where $G(\omega)$ is defined through

$$\begin{aligned} h_1(\tau) &\triangleq c_{4s}(0, \tau, \tau) + 2c_{2s}^2(\tau), \\ h_2(\tau) &\triangleq \bar{c}_{4v}(0, \tau, \tau) + 2\bar{c}_{2v}^2(\tau), \\ h_3(\tau) &\triangleq 4c_{2s}(\tau)\bar{c}_{2v}(\tau), \\ g_1(\omega) &\triangleq \frac{1}{8}H_1(2\omega) + \frac{1}{32}H_1(4\omega), \\ g_2(\omega) &\triangleq \frac{1}{2}H_2(2\omega), \\ g_3(\omega) &\triangleq \frac{1}{8}H_3(\omega) + \frac{1}{8}H_3(3\omega), \\ G(\omega) &\triangleq g_1(\omega) + g_2(\omega) + g_3(\omega). \end{aligned}$$

The most important aspect of these results is that the variance of the $\hat{\omega}_0$ estimate is still $O(T^{-3})$ despite of the fact that $v(t)$ is nonstationary. Although these derivations are for the cosinusoid model of (1), the $O(T^{-3})$ variance rate for $\hat{\omega}_0$ can be shown to apply for the complex harmonic model as well.

4. APPLICATION TO POLYNOMIAL PHASE SIGNALS

An M th-order polynomial phase signal (PPS) is modeled in discrete-time as [4, Sec. 12.6]

$$y(t) = A \exp\{j \sum_{m=0}^M a_m t^m\}, \quad j = \sqrt{-1}. \quad (27)$$

Although a PPS is generally aperiodic, its appropriately defined high-order instantaneous moment (HIM) can be periodic. Specifically, define for integer q , $y^{(*q)}(t) = y(t)$ if q is even and $y^*(t)$ if q is odd. Then for a fixed lag $\tau \neq 0$, the HIM operator is

$$\mathcal{P}_M[y(t); \tau] \triangleq \prod_{q=0}^{M-1} [y^{(*q)}(t - q\tau)]^{\binom{M-1}{q}}. \quad (28)$$

It can be shown that the application of (28) to (27) yields a pure harmonic at frequency $\tilde{\omega}_0 = M! \tau^{M-1} a_M$ [4, p. 395].

Now consider the PPS-in-noise process $x(t) = y(t) + v(t)$. The HIM operator applied to $x(t)$, $\mathcal{P}_M[x(t); \tau]$, will contain a harmonic at frequency $\tilde{\omega}_0$ plus a compounded additive noise process which contains not only $\mathcal{P}_M[v(t); \tau]$ but also a large number of cross terms produced by the nonlinear HIM operator. This compounded additive noise process is nonstationary even if $v(t)$ is stationary.

To estimate a_M , we simply take the FT of $\mathcal{P}_M[x(t); \tau]$ and then estimate $\tilde{\omega}$ from its peak location. When $\{x(t)\}_{t=0}^{T-1}$ is available, only $(T - (M-1)\tau)$ amount of data is utilized in forming $\mathcal{P}_M[x(t); \tau]$ according to (28). Therefore, using the variance expression developed in Section 3, we infer that the variance of $\hat{a}_M$ is proportional to $\tau^{-2(M-1)} (T - (M-1)\tau)^{-3}$. Let $g(\tau) = \tau^{2(M-1)} (T - (M-1)\tau)^3$. Then by setting $g'(\tau) = 0$, we find $\tau = T/(M+0.5)$ which is the optimum choice of lag that results in the minimum variance for the $\hat{a}_M$ estimate.

In [4, p. 398], it was mentioned that via extensive computer simulations, the optimum choice of τ that gives the lowest asymptotic variance for $\hat{a}_M$ was found to be $\tau \approx T/M$ for $M = 2, 3$ and $\tau \approx T/(M+2)$ for $4 \leq M \leq 10$. Our derivation for the optimum choice of τ corroborates these empirical results and is novel.

5. SIMULATIONS

In this section, we shall illustrate the performance of the proposed algorithms by way of computer simulations. We will see that the variance expressions (25) and (26) are valid regardless of the color and distribution of the multiplicative and additive noise processes.

Example 1: The nonzero mean case. We generated samples of $x(t)$ according to (1) with $\omega_0 = 1$. Multiplicative noise $s(t)$ was i.i.d. Gaussian with mean $m_s = 0.8$ and variance $\sigma_s^2 = 0.36$. Additive noise was $v(t) = d(t)\mu(t)$ where $d(t)$ was a deterministic function and $\mu(t)$ was an MA(2) process generated by passing zero-mean, unit-variance exponential variates through an FIR filter with coefficients [1, 0.5, 0.2]. Therefore, $v(t)$ is nonstationary. Because $s(t)$ is white, it has $S_{2s}(\omega) = 0.36, \forall \omega$.

First let $d(t) = \cos(0.7t^2)$. In this case,

$$\bar{S}_{2v}(\omega) = \sigma_\mu^2 \frac{1}{2} (1 + b_1^2 + b_2^2) = 0.645 \sigma_\mu^2,$$

$$\text{Var}(\hat{\omega}_0) = \frac{1}{T^3} \frac{15.48 \sigma_\mu^2 + 2.16}{0.64}. \quad (29)$$

In Figure 1(a), we show $|\hat{C}_{1x}(\alpha)|$ obtained with $T = 256$. Peaks were observed at $\alpha \approx \pm\omega_0$ and hence a search for a peak over $\alpha \in (0, \pi)$ yields an estimate for ω_0 . We zero-padded the data to length 2^{18} or 2^{20} prior to taking the FFT to ensure that the resolution of the frequency axis is fine enough. We used 100 independent realizations and calculated the variance of the resulting $\hat{\omega}_0$ estimates. Next, we varied T and set m_s^2/σ_μ^2 (by varying σ_μ^2) equal to -2dB (top), 8dB (middle), and 18dB (bottom) respectively, and plotted in Figure 1(b), the variance of the $\hat{\omega}_0$ estimates obtained from the Monte-Carlo runs (solid lines) and its corresponding value as given by (29) (dashed lines). We see that the large sample result (25) is valid even for T in the neighborhood of 100. Next, we used $d(t) = e^{-0.7t/T}$ and $d(t) = 1 + 0.7(t/T) + 0.5(t/T)^2$. The corresponding results are shown in Figures 1(c) and 1(d), respectively. Similar observations are made.

Example 2: The zero mean case. Both $s(t)$ and $\mu(t)$ are white Gaussian with zero-mean and unit variance. $d(t) = \cos(0.7t^2)$ was used. Figure 2(a) shows $|\hat{C}_{2x}(\alpha; 0)|$ obtained with $T = 256$ and is seen to peak around $\alpha = 0, \pm 2\omega_0$. Figure 2(b) shows the sample (solid lines) vs. theoretical (dashed lines) variance curves for $\hat{\omega}_0$ as σ_s^2/σ_v^2 varies from 0 to 10 to 20dBs (top to bottom): close agreement is observed between the empirical and the theoretical variance expressions for T above a reasonable threshold.

6. CONCLUSIONS

In this paper, we have studied the problem of harmonic retrieval in nonstationary additive noise and shown that FFT based algorithms still work as long as the additive noise process is not cyclostationary. Stationary multiplicative noise effects are also taken into account. We have developed variance expressions for the frequency estimates and seen that the same $O(T^{-3})$ variance rate applies. We carried out extensive computer simulations to examine the performance of the algorithms and gave a polynomial phase signal example to illustrate the applicability of the results.

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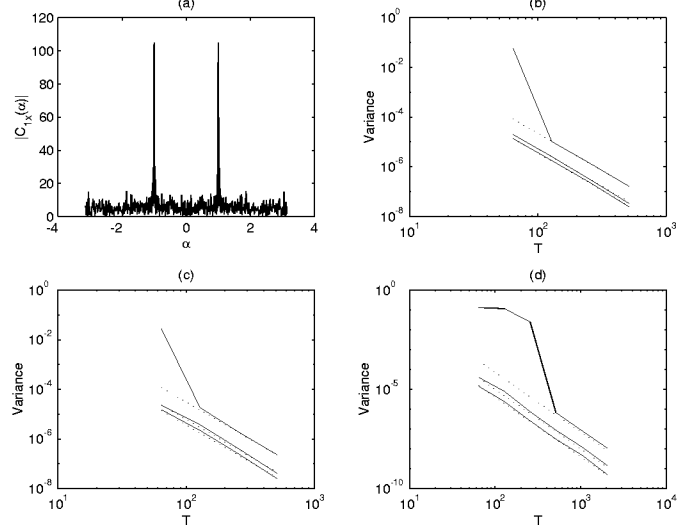


Figure 1. The nonzero mean case.

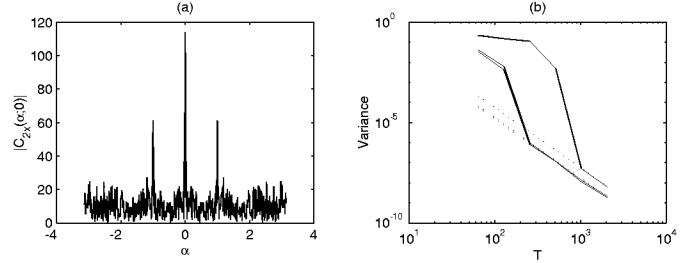


Figure 2. The zero mean case.

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