

# IDENTIFIABILITY AND MANIFOLD AMBIGUITY IN DOA ESTIMATION FOR NONUNIFORM LINEAR ANTENNA ARRAYS

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## ABSTRACT

This paper considers the direction-of-arrival (DOA) estimation identifiability problem for uncorrelated Gaussian sources and non-uniform antenna arrays. It is now known that sparse arrays always suffer from *manifold ambiguity*, which arises due to linear dependence amongst the columns of the array manifold matrix (the “steering vectors”). While the standard subspace DOA estimation algorithms such as MUSIC fail to provide proper unambiguous estimates under these conditions, we demonstrate that in most cases involving uncorrelated Gaussian sources, manifold ambiguity does not necessarily imply nonidentifiability. An effective manifold ambiguity resolution algorithm is introduced. A *superior* number of uncorrelated Gaussian sources (more than sensors) may also be unambiguously localised by sparse arrays under specified identifiability conditions. While manifold ambiguity does not apply to superior scenarios, a similar “co-array manifold ambiguity” phenomenon may compromise DOA estimation. The proposed algorithm can also resolve such ambiguity in all identifiable cases.

## 1. INTRODUCTION

The problem of specifying the conditions under which DOA estimation of narrow-band sources has a unique solution is “of crucial importance” [10]. In [9], the conditions that specify the maximum number of sources that can be uniquely localised are found in terms of the number of sensors ( $M$ ) and the rank of the intersensor correlation matrix, under the assumption that “any subset of  $M$  distinct steering vectors from the array manifold is linearly independent”. It was mentioned in [9, 10] that this assumption “imposes certain constraints on the array geometry”. At the same time, the implications of such exceptions have not been fully recognised due to the perception that “these constraints, however, do not pose a serious problem and can easily be come by” [10].

Recent results obtained by Proukakis and Manikas [7] have proven that these constraints do pose a serious practical problem for all nonuniform linear arrays (NLA's) mentioned in the literature. They demonstrated that a sufficient condition for the presence of manifold ambiguity in *any* linear array is

$$d_M > M - 1 \quad (1)$$

where  $d_M$  is the array's aperture measured in half-wavelengths. Moreover, they introduced an approach to calculate the “ambiguous generator sets” (AGS's) of any NLA, *ie.* the sets of DOA's

that correspond to linearly dependent steering vectors. By virtue of (1), *all half-wavelength sparse integer linear arrays are manifoldly ambiguous*, and therefore traditional subspace techniques (*eg.* MUSIC) fail to properly identify sources with a rank-deficient array manifold matrix. Even so, this failure is algorithm-specific and does not imply nonidentifiability; for uncorrelated sources, this fact is quite evident for *fully-augmentable* NLA's [2] which may be treated by the direct augmentation approach (DAA) introduced by Pillai *et al.* in [6] (see [2] for details).

Moreover, the identifiability conditions defined in [10] for unambiguous scenarios uses an *arbitrary* intersource correlation matrix of a given rank, and consequently cannot be applied beyond the “conventional” number of sources,  $m < M$ .

In [9], the class of signals was later restricted “to certain loci in the complex plane”, which almost doubled the maximum number of identifiable sources. This makes it clear that the *a priori* statistical model of the signals is critically important, *as well as* the antenna array geometry.

For these reasons, we investigate the correspondence between identifiability and manifold ambiguity in sparse arrays for some important Gaussian source models. For all situations where manifold ambiguity does not lead to nonidentifiability, we need an appropriate technique to properly resolve manifold ambiguity. Lastly, since NLA's permit DOA estimation beyond the limit of the “conventional case”, we need to comment on the issue of identifiability in the “superior case” ( $m \geq M$ ), where manifold ambiguity loses its original meaning, but a similar phenomenon occurs.

## 2. BACKGROUND: MANIFOLD AMBIGUITY, LOCAL AND POINTWISE NONIDENTIFIABILITY

Consider an  $M$ -element NLA, with sensors located at positions  $\mathbf{d} \equiv [d_1, \dots, d_M]$  measured in half-wavelength units; and set  $d_1 = 0$  for convenience. The problem of DOA estimation for  $m$  Gaussian plane wave sources impinging on a linear antenna array consisting of  $M$  identical omnidirectional sensors can be reduced to estimation of the unknown azimuthal angle parameter  $\boldsymbol{\theta} \equiv [\theta_1, \dots, \theta_m]^T$  in the equation

$$\mathbf{y}(t) = S(\boldsymbol{\theta}) \mathbf{x}(t) + \mathbf{n}(t) \quad \text{for } t = 1, \dots, N \quad (2)$$

where  $\mathbf{y}(t) \in \mathcal{C}^{M \times 1}$  is the column vector of array sensor outputs observed at time  $t$  (the “snapshot”),  $S(\boldsymbol{\theta}) \in \mathcal{C}^{M \times m}$  is the array manifold matrix,  $\mathbf{x}(t) \in \mathcal{C}^{m \times 1}$  is the vector of Gaussian signal amplitudes, and  $\mathbf{n}(t)$  is additive noise. We assume this noise is

\*This study was partly supported by the INTAS SASPARC grant.

white and Gaussian, with known power  $\sigma$ :

$$\mathcal{E}\{\mathbf{n}(t_1)\mathbf{n}^H(t_2)\} = \begin{cases} \sigma I_M & \text{for } t_1 = t_2 \\ 0 & \text{for } t_1 \neq t_2, \end{cases} \quad (3)$$

where  $\mathcal{E}\{\cdot\}$  is the expectation operator. Note that the number of sources  $m$  is supposed to be known *a priori*. The array manifold matrix is  $S(\boldsymbol{\theta}) \equiv [s(\theta_1), \dots, s(\theta_m)]$  where each

$$s(\theta_j) = \left[ 1, \exp(i\pi d_2 \sin \theta_j), \dots, \exp(i\pi d_M \sin \theta_j) \right]^T \quad (4)$$

is a so-called steering vector. Traditionally, it has been assumed that the manifold matrix  $S(\boldsymbol{\theta})$  is of full (column) rank (eg. [8, 9, 10]). With NLA's, we must abandon this assumption, due to (1).

Recall that the *co-array* of a linear array ( $\mathcal{D}'$ ) is the sorted set of nonduplicated elements of  $\mathcal{D}$  (the set of all intersensor differences), thus the number of co-array elements is

$$M_{CA} = \frac{1}{2}M(M-1) + 1 - \mathcal{R} \quad (5)$$

where  $\mathcal{R}$  is the number of covariance lag redundancies.

**Definition 1** Let  $y_1, \dots, y_m$  be generated by a parametric model  $\mathcal{M}_\theta$ , with  $\theta \in \Theta$ , characterized by some p.d.f.  $f(\mathbf{y}; \theta)$ . Then the parameter  $\theta$  is *nonidentifiable* at  $\theta_* \in \Theta$  if there exists  $\theta'_* \neq \theta_* \in \Theta$  such that

$$f(\mathbf{y}; \theta_*) = f(\mathbf{y}; \theta'_*) \quad \text{almost surely (a.s.).} \quad (6)$$

It is important to note that nonidentifiability may not preclude local identifiability (ie. consistency and asymptotic efficiency of the ML estimate within some open set  $\mathcal{O} \ni \theta_*$ , where  $\theta_*$  is the “true” value). This is because nonidentifiability may exist at isolated points of the set  $\Theta$ , while traditional regularity conditions are defined *locally*. If (6) is satisfied at one or more isolated points in  $\Theta$ , then we call this *pointwise nonidentifiability*.

**Definition 2** Let  $\mathcal{M}_\theta$  be a parametric model, with  $\theta \in \Theta$ . Then the parameter  $\theta$  is *locally nonidentifiable* at  $\theta_* \in \Theta$  if, for any open set  $\mathcal{O} \subset \Theta$  such that  $\theta_* \in \mathcal{O}$  and any  $\varepsilon > 0$ , there exists a  $\theta_0 \in \mathcal{O}$  such that the model  $\mathcal{M}_\theta$  is Cramér–Rao regular at  $\theta_0$ , with Fisher information matrix (FIM)  $F_{\theta_0}$  satisfying

$$0 < F_{\theta_0} < \varepsilon I_\ell \quad (7)$$

where  $I_\ell$  is the  $\ell \times \ell$  identity matrix and  $\ell$  is the parameter size.

**Theorem 1** Let  $\mathcal{M}_\theta$  be a parametric model with a continuously differentiable function  $\log f(\mathbf{y}; \theta)$  and a continuous FIM  $F_\theta$ , both at  $\theta_* \in \Theta$ . Assume that  $\mathcal{M}_\theta$  satisfies the standard regularity conditions, except for the strict positivity of  $F_\theta$ , and let  $F_{\theta_*}$  be singular. Then  $\theta$  is nonidentifiable (at least locally) at  $\theta_*$ .

For proof, see [4]. The meaning of Definition 2 and Theorem 1 is that, given a singular FIM (and some other technical conditions stated above), there are only two possibilities:

1. the DOA estimation variance may be arbitrarily large in the vicinity of the “bad point” (CRB  $\rightarrow \infty$ ), or
2. the model is nonidentifiable in some open set which contains this “bad point”.

In (1), any obviously uncontrollable variance means in practice a lack of consistency.

Given these definitions, it is straight-forward to demonstrate that the manifold ambiguity condition coincides with the pointwise nonidentifiability condition for the following two Gaussian models:

- Conditional Model Assumption (CMA) [8]:

$$\mathbf{y}(t) \sim \mathcal{CN}\left(S(\boldsymbol{\theta})\mathbf{x}(t), \sigma I_M\right)$$

- rank-one Unconditional Model Assumption (UMA) [8]:

$$\mathbf{y}(t) \sim \mathcal{CN}\left(0, \sigma I_M + p S(\boldsymbol{\theta})\mathbf{b}\mathbf{b}^H S^H(\boldsymbol{\theta})\right).$$

Indeed, since  $\mathbf{x}(t)$  and  $\mathbf{b}$  are arbitrary complex vectors, pointwise nonidentifiability means that for some given  $\{\theta_*, \mathbf{x}_*(t)\}$  (for CMA, or  $\{\theta_*, \mathbf{b}_*\}$  for rank-one UMA) there exists another set  $\{\theta'_*, \mathbf{x}'_*(t)\}$  (or  $\{\theta'_*, \mathbf{b}'_*\}$ ) with at least one  $\theta'_{*j}$  different from  $\theta_{*j}$ :

$$S(\theta_*)\mathbf{x}_*(t) = S(\theta'_*)\mathbf{x}'_*(t) \quad \text{or} \quad S(\theta_*)\mathbf{b}_* = S(\theta'_*)\mathbf{b}'_* \quad (8)$$

Suppose that  $\theta'_*$  differs from  $\theta_*$  at a single DOA, so that the set  $\tilde{\theta}_* = \theta_* \cup \theta'_*$  consists of  $(m+1)$  points, with  $S(\tilde{\theta}_*) \in \mathcal{C}^{M \times (m+1)}$  being the manifold matrix for all  $(m+1)$  points. Then the pointwise nonidentifiability condition (8) exists if and only if the matrix  $S(\tilde{\theta}_*)$  is (column) rank-deficient, ie.

$$\text{rank } S(\tilde{\theta}_*) \leq m. \quad (9)$$

This single necessary and sufficient condition is exactly the manifold ambiguity condition studied by Proukakis and Manikas [7]. Since  $S(\tilde{\theta}_*)$  is an  $M \times (m+1)$  matrix, it is clear that for these two signal models, *any*  $m \geq M$  sources are always pointwise nonidentifiable, while for linear arrays with  $m < M$  sources that satisfy the sufficient condition for manifold ambiguity ( $d_M > M-1$ ), there will always exist an ambiguous DOA set that satisfies (9).

For the general UMA model, which involves an *arbitrary* positive-definite matrix  $B$  and a distribution

$$\mathbf{y}(t) \sim \mathcal{CN}(0, \sigma I_M + S(\boldsymbol{\theta}) B S^H(\boldsymbol{\theta})), \quad (10)$$

the nonidentifiability condition means that

$$S(\theta_*) B_* S^H(\theta_*) = S(\theta'_*) B'_* S^H(\theta'_*) \quad (11)$$

and for a covariance matrix  $B > 0$  that is not restricted to any specific class, one can prove that (9) is sufficient for the nonidentifiability condition (11) to take place. Thus manifold ambiguity always implies statistical nonidentifiability for this UMA model.

Let us now concentrate on the model (10) with uncorrelated sources, where  $B = \text{diag}[p_1, \dots, p_m] > 0$ . According to Definition 1, the parameters  $\theta_*$  and  $p_*$  are pointwise nonidentifiable if there exists another set of DOA's  $\theta_* \neq \theta'_*$ ,  $\theta_* \in \Theta$  and a set of powers  $p'_* > 0$  such that

$$\sum_{j=1}^m p_{*j} S(\theta_{*j}) S^H(\theta_{*j}) = \sum_{j=1}^m p'_{*j} S(\theta'_{*j}) S^H(\theta'_{*j}). \quad (12)$$

If only the DOAs  $\theta_*$  are specified, then nonidentifiability occurs when we can find the three sets  $p_* > 0$ ,  $p'_* > 0$  and  $\theta'_*$  that satisfy (12). In other words, for some specific set of powers  $p_*$ , any given scenario may well be identifiable, whereas when these powers are not specified by the *a priori* model, there may well be a particular set of powers  $p_*$  and  $p'_*$  that results in nonidentifiability.

For fully-augmentable arrays (integer NLA's where  $\mathcal{D}$  is complete, ie. all  $M_\alpha$  contiguous covariance lags are present), the associated co-arrays are always uniform with  $M_\alpha$  elements. Using the properties of Vandermonde matrices, it is straight-forward to show that under this uncorrelated signal model, any scenario with

$$M \leq m < M_\alpha \quad (13)$$

sources is identifiable, regardless of manifold ambiguity, and indeed that existing subspace-type techniques fail in these cases.

For partially-augmentable *maximum-contiguous-lag* arrays [1], we have

$$M \ll N_{\max} \leq \frac{1}{2}M(M-1) \quad (14)$$

(where  $N_{\max}$  is the greatest multiple of the unit spacing such that all lags up to  $N_{\max}$  inclusive are present), these conclusions are also valid provided that  $m < N_{\max}$ .

A less trivial problem regarding identifiability and manifold ambiguity conditions exists for partially-augmentable arrays with an “early” gap. Let the number of gaps be  $\mathcal{G}$ , and let  $\mathbf{r}$  be the  $(M_\alpha - \mathcal{G})$ -variate vector containing all distinct spatial covariance lags for this NLA:  $\mathbf{r} = [\{r_j\}]$ ,  $r_j \in \mathcal{D}$ . Then

$$\mathbf{r} = \mathcal{A}(\boldsymbol{\theta}_*) \mathbf{p}_* \quad (15)$$

where  $\mathcal{A}(\boldsymbol{\theta}_*) \in \mathcal{C}^{(M_\alpha - \mathcal{G}) \times m}$  is the co-array manifold matrix. Since the Vandermonde properties are lost in  $\mathcal{A}(\boldsymbol{\theta}_*)$ , finding the identifiability condition is not trivial. In principle, there could be solutions of the system

$$\mathcal{A}(\boldsymbol{\theta}_*) \mathbf{p}_* = \mathcal{A}(\boldsymbol{\theta}'_*) \mathbf{p}'_* \quad \text{for } \boldsymbol{\theta}_* \neq \boldsymbol{\theta}'_*, \quad \mathbf{p}_*, \mathbf{p}'_* > 0. \quad (16)$$

Let us firstly define the necessary conditions for such a solution to exist. As before, suppose that at least one of the DOA's in the set  $\boldsymbol{\theta}'_*$  is different from the set  $\boldsymbol{\theta}_*$ . Then  $\tilde{\boldsymbol{\theta}}_* = \boldsymbol{\theta}_* \cup \boldsymbol{\theta}'_*$  consists of  $(m+1)$  points, and the co-array manifold ambiguity condition

$$\text{rank } \mathcal{A}(\tilde{\boldsymbol{\theta}}_*) \leq m \quad (17)$$

becomes the first evident necessary condition. Since the elements of the vectors  $\mathbf{p}_*$  and  $\mathbf{p}'_*$  are real numbers, (16) can be rewritten as

$$\mathcal{A}_r(\tilde{\boldsymbol{\theta}}_*) \mathbf{a} = 0 \quad (18)$$

where

$$\mathcal{A}_r(\tilde{\boldsymbol{\theta}}_*) = \begin{bmatrix} \text{Re } \mathcal{A}(\tilde{\boldsymbol{\theta}}_*) \\ \text{Im } \mathcal{A}(\tilde{\boldsymbol{\theta}}_*) \end{bmatrix} \in \mathcal{R}^{(2(M_\alpha - \mathcal{G}) - 1) \times (m+1)} \quad (19)$$

(the first row of the imaginary part is dropped, since it is trivially equal to zero), and

$$\mathbf{a} = \begin{bmatrix} p_{*1} \\ p_{*2} - p'_{*2} \\ \vdots \\ p_{*m} - p'_{*m} \\ -p'_{*m+1} \end{bmatrix} \quad \text{for } \mathbf{p}_*, \mathbf{p}'_* > 0. \quad (20)$$

Thus a nonzero solution of this system exists only if the real matrix  $\mathcal{A}_r(\tilde{\boldsymbol{\theta}}_*)$  is rank-deficient:

$$\text{rank } \mathcal{A}_r(\tilde{\boldsymbol{\theta}}_*) \leq m. \quad (21)$$

Clearly this second necessary condition is significantly different from the manifold ambiguity condition  $\text{rank } S(\tilde{\boldsymbol{\theta}}_*) \leq m$ , and from the co-array manifold ambiguity condition  $\text{rank } \mathcal{A}(\tilde{\boldsymbol{\theta}}_*) \leq m$ .

In general, the AGS  $\tilde{\boldsymbol{\theta}}_*$  may consist of an arbitrary number of directions  $m+1 \leq \ell \leq 2m$ , while the rank of the matrix  $\mathcal{A}_r(\tilde{\boldsymbol{\theta}}_*)$  could be  $(m-\kappa)$  for  $0 \leq \kappa \leq m-2$ . Then the general sufficient condition for some fixed set of source powers  $\mathbf{p}_*$  is

$$\Phi \mathbf{x} + \mathbf{z} = \begin{bmatrix} \mathbf{p}_* \\ 0 \end{bmatrix} \quad \begin{matrix} m \\ \ell - m \end{matrix} \quad (22)$$

where  $\Phi \in \mathcal{R}^{\ell \times (\ell - m + \kappa)}$  is the fundamental solution to the system (18), and  $\mathbf{z} = [0, \dots, 0 \mid z_{\ell-m+1}, \dots, z_\ell]^T$  with  $z_j \geq 0$ . The solution to (22) may be found by solving the following linear programming (LP) problem:

$$\min \sum_{j=1}^{\ell} (\lambda_{1j} + \lambda_{2j}) \quad \text{for } \Phi(\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{z} + \lambda_1 - \lambda_2 = \begin{bmatrix} \mathbf{p}_* \\ 0 \end{bmatrix} \quad (23)$$

where  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{R}_+^{(\ell-m+\kappa) \times 1}$  and  $\lambda_1, \lambda_2 \in \mathcal{R}_+^{\ell \times 1}$  (ie. positive real numbers). The necessary and sufficient conditions (18) and (20) are satisfied if this minimum is zero.

Thus, for any Gaussian scenario  $\boldsymbol{\theta}_*$  with uncorrelated sources and rank-deficient matrix  $\mathcal{A}_r(\boldsymbol{\theta}_*)$ , nonidentifiability occurs if the solution of the LP problem (23) is equal to zero. Note that this condition is also true for the superior case ( $M \leq m < M_\alpha$ ). Also, with some obvious modifications to  $\mathcal{A}_r(\tilde{\boldsymbol{\theta}}_*)$ , the same procedure and conclusion may be obtained for noninteger arrays.

When the set of source powers is arbitrary, the corresponding LP problem may be formulated as follows:

$$\min \sum_{k=1}^4 \sum_{j=1}^{\ell-m} \lambda_{kj} \quad \text{for } \Phi(\mathbf{x}_1 - \mathbf{x}_2) + \begin{bmatrix} \mathbf{0} \\ \mathbf{z}'_1 \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{z}'_2 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} -\mathbf{z}''_1 \\ \mathbf{0} \\ \mathbf{z}''_2 \end{bmatrix} + \begin{bmatrix} \lambda_{11} \\ \mathbf{0} \\ \lambda_{1,m+1} \end{bmatrix} - \begin{bmatrix} \lambda_{21} \\ \mathbf{0} \\ \lambda_{2,m+1} \end{bmatrix} = 0 \quad (24)$$

where  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{R}_+^{(\ell-m+\kappa) \times 1}$ ;  $\mathbf{z}'_1, \mathbf{z}'_2 \in \mathcal{R}_+^{(2m-\ell) \times 1}$ ; and  $\mathbf{z}''_1, \mathbf{z}''_2, \lambda_k \in \mathcal{R}_+^{(\ell-m) \times 1}$ .

The above necessary (17) and (21) and sufficient conditions (ie. minimum of (23) or (24) is zero) for nonidentifiability allow us to analyze any manifold ambiguity generator set (AGS) [7].

Moreover, the Proukakis–Manikas AGS-finding method could also be useful in searching for pointwise nonidentifiable superior scenarios. Indeed, by applying their algorithm to the co-array so as to find AGS's that satisfy (17), we may search amongst these sets for those that meet the necessary (21) and sufficient conditions (23) and (24). Note that if such pointwise nonidentifiable scenarios exist, they may consist of isolated DOA points, ie. for a given  $\boldsymbol{\theta}_*$ , only a finite number of different sets  $\boldsymbol{\theta}'_*$  may be found. Once again, this means that in the vicinity of the true DOA's  $\boldsymbol{\theta}_*$ , ML estimation may well be locally consistent. Nevertheless, for the admissible number of sources  $m < M_{CA}$ , it is possible to find some scenarios that are not identifiable even *locally*, that is, in an arbitrarily small neighborhood of the true DOAs.

This local nonidentifiability phenomenon is illustrated by the  $M = 5$ ,  $m = 6$  scenario for the *minimum-gaps* (Golomb) array:

$$\mathbf{d}_{PA} = [0, 1, 4, 9, 11], \quad \mathbf{w} = \left[-\frac{5}{6}, -\frac{1}{2}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right] \quad (25)$$

This scenario gives rise to a diagonal 5-variate covariance matrix  $R$ , which is invariant under angle shifts in all six directions.

### 3. NONAMBIGUOUS DOA IDENTIFICATION BY MODEL-FITTING

For a conventional number of uncorrelated Gaussian sources, MUSIC applied to  $\hat{R}$  gives  $(m+\kappa)$  DOA estimates,  $\kappa > 0$ , for every manifoldly ambiguous scenario (9). In the corresponding superior case, similar co-array ambiguity occurs for every scenario satisfying the co-array manifold ambiguity condition (17), when MUSIC

is applied to the augmented  $M_{CA}$ -variate covariance matrix of the virtual array (the co-array in this case). The details of this generalised augmentation approach (GAA) are discussed elsewhere [1, 3, 5]. Here we formulate a method which rules out the  $\kappa$  spurious DOA's by matching the overall signal model, provided that it is known to be identifiable.

Let  $\mathcal{A}(\hat{\theta}_*) \in \mathcal{C}^{M_{CA} \times (m+\kappa)}$  be the co-array manifold matrix corresponding to the manifoldly ambiguous set of DOA estimates  $\hat{\theta}_*$ . Let  $\hat{r}$  be the  $M_{CA}$ -variate vector of all different covariance lags, estimated using the direct data covariance (DDC) matrix  $\hat{R}$  and the DAA. Recall that for the absolutely accurate parameters  $\theta_*$  and  $r$ , identifiability means that the linear equation

$$\mathcal{A}(\tilde{\theta}_*)p = r \quad \text{with } p \in \mathcal{R}^{m+\kappa} \quad (26)$$

has precisely  $m$  positive entries corresponding to the true DOA's and precisely  $\kappa$  zero entries corresponding to the spurious DOA's.

Thus for the estimated parameters  $\hat{\theta}_*$  and  $\hat{r}$ , we may search for the best fit over all  $(m+\kappa)$  entries of the constrained solution  $\hat{p} \geq 0$ , then to associate the  $m$  greatest powers with the true DOA's.

We solve this ‘‘diagonal fit’’ problem (DFP) with a LP method, the details of which are presented in [4]. The solution of the DFP does not depend on the array geometry, and could be easily generalized for arbitrary nonlinear arrays (circular, 2-D, *etc.*), provided that identifiability is guaranteed for the given scenario.

To demonstrate the efficiency of manifold ambiguity resolution by diagonal fitting, we consider the noninteger array [7]

$$d_{NI} = [0, 1.2, 3.4, 4.6] \quad (27)$$

whose co-array is

$$d'_{NI} = [0, 1.2, 2.2, 3.4, 4.6] \quad (28)$$

and hence only up to four sources could be identifiable. For this reason, we are only able to explore (up to) three-source conventional scenarios, so we choose the following subset of one particular AGS for our simulations:

$$w_3 = [-0.8391, -0.4043, 0.4652]. \quad (29)$$

We perform 1000 random trials where the sources have equal power and a SNR of 20 dB. Table 1 shows the probability of correct identification for various sample sizes. Apart from the very high success rates achieved, it is appropriate to comment that since this scenario has widely separated sources, all trials were ‘‘regular’’, except at  $N = 10$  where MUSIC failed to detect the four peaks in only 11 trials. By ‘‘regular’’, we mean that (a) MUSIC( $\hat{R}$ ) gave at least  $(m+\kappa)$  peaks, and (b) these peaks were *normal* DOA estimates [2] (*ie.* no main peaks were merged).

The consistency of these results are shown by the second simulation results over 10000 trials, which is able to be quickly performed due to the very efficient implementation in standard computing environments of LP routines.

Snapshots ( $N$ )	10	20	50	100	500
$P$ (1000 trials)	0.691	0.820	0.958	0.995	1.000
$P$ (10000 trials)	0.665	0.803	0.948	0.992	1.000

Table 1: Sample probability ( $P$ ) convergence for correct identification by diagonal fitting for the non-integer array  $d_{NI}$ .

## 4. SUMMARY

We have demonstrated that manifold ambiguity defined via the linear dependence of the NLA array manifold (‘‘steering’’) vectors does not necessarily mean that the corresponding scenario is non-identifiable. The failure of subspace algorithms, such as MUSIC, to give unambiguous DOA estimates means that these methods alone are inadequate, and need to be substituted or complemented by some other techniques capable of manifold ambiguity resolution. In this paper we have introduced one approach to manifold ambiguity resolution for identifiable uncorrelated Gaussian sources. This method seeks the best fit amongst the set of estimated spatial covariance lags and source powers for each of the MUSIC DOA estimates, including the ambiguous ones. This proposed fitting procedure adopts a computationally efficient linear programming routine and demonstrates an extremely high probability of correct identification in manifoldly ambiguous scenarios. We have also demonstrated that local nonidentifiability can occur for partially-augmentable arrays with a superior number of sources.

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