GABOR'S SIGNAL EXPANSION ON A QUINCUNX LATTICE AND THE MODIFIED ZAK TRANSFORM

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ABSTRACT

Gabor's expansion of a signal on a quincunx lattice with oversampling by a rational factor is presented for continuous-time signals. It is shown how a modified Zak transform instead of the ordinary Zak transform can be helpful in determining Gabor's signal expansion coefficients and how it can be used in finding the dual window. Furthermore, some examples of dual windows for the quincunx case are given and compared with dual windows for the rectangular case.

1. INTRODUCTION

The case of Gabor's signal expansion on a rectangular lattice and its connection with the Zak transform has been studied extensively (see, e.g., [1] and [2]). Recently, the connection between Gabor's signal expansion on a quincunx lattice, a sampling geometry which is different from the traditional rectangular sampling geometry (see Fig. 1), and the ordinary Zak transform has been shown [3]. In the

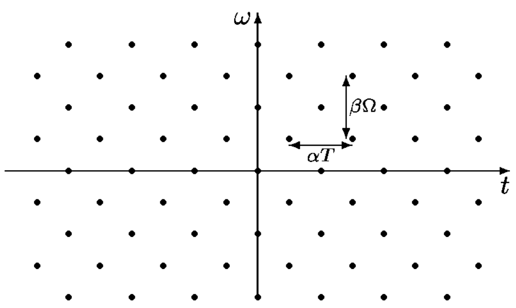

Figure 1: The quincunx lattice

case of critical sampling one obtains then a sum-of-products form instead of a simple product form which appears in the rectangular case. It is also possible to use a modified Zak transform. With this modified Zak transform one obtains then in the case of critical sampling a product form and the results are therefore comparable with the rectangular case [4]. In this paper we extend this idea of the connection between the Gabor transform and the modified Zak transform to the case of oversampling with a rational factor.

2. GABOR'S SIGNAL EXPANSION

The discrete set of shifted and modulated versions of the elementary signal $g(t)$ on a rectangular lattice reads

$$g_{mk}(t) = g(t - m\alpha T)e^{jk\beta\Omega t} \tag{1}$$

where the time shift αT and the frequency shift $\beta\Omega$ satisfy the relationships $\Omega T = 2\pi$ and $\alpha\beta = q/p \leq 1$, where p and q are positive integers, $p \geq q \geq 1$, and where m and k may take all integer values. Gabor's expansion of a signal $\varphi(t)$ into a discrete set of shifted and modulated versions of an elementary signal $g(t)$ on a rectangular lattice reads

$$\varphi(t) = \sum_{mk} a_{mk} g_{mk}(t), \tag{2}$$

where

$$a_{mk} = <\varphi, \gamma_{mk}> = \int \varphi(t)\gamma^*_{mk}(t)\,dt \tag{3}$$

with $\gamma_{mk}(t)$ the shifted and modulated versions of the dual window $\gamma(t)$ [cf. (1)]. Combining (2) and (3) yields the condition [2]

$$\sum_m g(t - m\alpha T)\gamma^*\left(t + k\frac{T}{\beta} - m\alpha T\right) = \frac{\beta}{T}\delta[k], \tag{4}$$

where $\delta[k]$ is a Kronecker delta, with $\delta[0] = 1$ and $\delta[k] = 0$ for $k \neq 0$.

The Fourier transform $\bar{a}(x, y)$ of an array a_{mk} is defined according to

$$\bar{a}(x, y) = \sum_m \sum_k a_{mk} e^{-j2\pi(my - kx)} \tag{5}$$

and the Zak transform $\tilde{\varphi}(t,\omega;\tau)$ of a signal $\varphi(t)$ is defined as

$$\tilde{\varphi}(t,\omega;\tau)=\sum_m \varphi(t+m\tau)e^{-jm\omega\tau}. \quad (6)$$

Using the Fourier transform and the Zak transform, it was shown that (2) and (3) can be transformed into the sum-of-products forms [2]

$$\tilde{\varphi}\left((x+s)\frac{\alpha pT}{q},y\frac{\Omega}{\alpha};\alpha pT\right)=\frac{1}{p}\sum_{r=<p>}$$

$$\bar{a}\left(x,y+\frac{r}{p}\right)\tilde{g}\left((x+s)\frac{\alpha pT}{q},\left[y+\frac{r}{p}\right]\frac{\Omega}{\alpha};\alpha T\right), \quad (7)$$

and

$$\bar{a}\left(x,y+\frac{r}{p}\right)=\frac{\alpha pT}{q}\sum_{s=<q>}$$

$$\tilde{\varphi}\left((x+s)\frac{\alpha pT}{q},y\frac{\Omega}{\alpha};\alpha pT\right)$$

$$\times\tilde{\gamma}^*\left((x+s)\frac{\alpha pT}{q},\left[y+\frac{r}{p}\right]\frac{\Omega}{\alpha};\alpha T\right), \quad (8)$$

respectively, where x extends over an interval of length 1 and y over an interval of length $1/p$. The expressions $s=<q>$ and $r=<p>$ are used as short-hand notations for an interval of q and p successive integers, respectively.

3. GABOR'S SIGNAL EXPANSION ON A QUINCUNX LATTICE

In the case of a quincunx lattice (see Fig. 1) we still have (2) and (3) but now we have a different discrete set of shifted and modulated versions of $g(t)$ and $\gamma(t)$

$$g_{mk}(t)=\tfrac{1}{2}(1+(-1)^{m+k})g(t-\tfrac{1}{2}m\alpha T)e^{jk\frac{1}{2}\beta\Omega t}$$

$$\gamma_{mk}(t)=\tfrac{1}{2}(1+(-1)^{m+k})\gamma(t-\tfrac{1}{2}m\alpha T)e^{jk\frac{1}{2}\beta\Omega t}. \quad (9)$$

The time shift $\frac{1}{2}\alpha T$ and the frequency shift $\beta\Omega$ satisfy the relationships $\Omega T=2\pi$ and $\frac{1}{2}\alpha\beta=q/p\leq 1$, where p and q are positive integers, $p\geq q\geq 1$.

After some manipulation we get the condition [cf. (4)]

$$\sum_m(-1)^{mk}g(t-\tfrac{1}{2}m\alpha T)\gamma^*(t-k\frac{T}{\beta}-\tfrac{1}{2}m\alpha T)=\frac{\beta}{T}\delta[k]. \quad (10)$$

At this point we write

$$(-1)^{mk}=e^{j\frac{\pi}{2pq}(mq+kp)^2}e^{-j\frac{\pi}{2pq}m^2q^2}e^{-j\frac{\pi}{2pq}k^2p^2}$$

and make use of a modified Zak transform $\widehat{\varphi}(t,\omega;\tau,C)$ of a signal $\varphi(t)$, with C a constant equal to 1 or p, defined as [cf. (6)]

$$\widehat{\varphi}(t,\omega;\tau,C)=\sum_m\varphi(t+m\tau)e^{-j\frac{q\pi}{2p}(Cm)^2}e^{-jm\omega\tau}. \quad (11)$$

It has been shown [4] that in the case of critical sampling the following product can be derived from the condition (10)

$$T\,\widehat{g}(x\frac{T}{\beta},y\frac{2\Omega}{\alpha};\tfrac{1}{2}\alpha T,1)\widehat{\gamma}^*(x\frac{T}{\beta},y\frac{2\Omega}{\alpha};\tfrac{1}{2}\alpha T,1)=\beta,$$

with $\alpha\beta=2$, which is almost identical with the well-known product form of the rectangular case as already mentioned. It is also possible to use the ordinary Zak transform; however then we obtain a sum-of-products form instead of a product form [3].

We remark that, compared to the ordinary Zak transform (6), an additional phase factor $\exp[-j\frac{q\pi}{2p}(Cm)^2]$ arises in the definition of the modified Zak transform (11). Moreover, we remark that the modified Zak transform $\widehat{\varphi}(t,\omega;\tau,C)$ is periodic in the frequency variable ω with period $\Omega=2\pi/\tau$ and quasi-periodic in the time variable t with a period which depends on q: in particular we have to treat the cases q is even and odd differently.

In the case that q is even, we can derive from this condition (10) the matrix product

$$\boldsymbol{G}^e\boldsymbol{\Gamma}^{e*}=\frac{p\beta}{T}\boldsymbol{I}_q, \quad (12)$$

with $\boldsymbol{G}^e$ a $q\times p$ matrix with elements

$$G^e_{ik}=\widehat{g}\left((2x+i)\frac{T}{\beta},(y+k/p+i/2)\frac{2\Omega}{\alpha};\tfrac{1}{2}\alpha T,1\right),$$

with $\boldsymbol{\Gamma}^e$ a $q\times p$ matrix with elements

$$\Gamma^e_{ik}=\widehat{\gamma}\left((2x+i)\frac{T}{\beta},(y+k/p+i/2)\frac{2\Omega}{\alpha};\tfrac{1}{2}\alpha T,1\right),$$

and with $\boldsymbol{I}_q$ the $q\times q$ identity matrix.

In the case that q is odd we have

$$\boldsymbol{G}^o\boldsymbol{\Gamma}^{o*}=\frac{2p\beta}{T}\boldsymbol{I}_{2q}, \quad (13)$$

with $\boldsymbol{G}^o$ a $2q\times 2p$ matrix with elements

$$G^o_{ik}=\widehat{g}\left((2x+i)\frac{T}{\beta},(y+k/2p+i/2)\frac{2\Omega}{\alpha};\tfrac{1}{2}\alpha T,1\right)$$

and $\boldsymbol{\Gamma}^o$ a $2q\times 2p$ matrix with elements

$$\Gamma^o_{ik}=\widehat{\gamma}\left((2x+i)\frac{T}{\beta},(y+k/2p+i/2)\frac{2\Omega}{\alpha};\tfrac{1}{2}\alpha T,1\right).$$

By using the Fourier transform, as defined in (5), of the array a_{mk} multiplied by an additional phase factor $\exp(j\frac{q\pi}{2p}m^2)$ and by using the ordinary and modified Zak transforms, as defined in (6) and (11), it can be shown that

$$\bar{a}(x,y)=\sum_{mk}\left[a_{mk}e^{j\frac{q\pi}{2p}m^2}\right]e^{-j2\pi(my-kx)}$$

can be transformed into the sum-of-products forms

$$\begin{aligned}&\overline{a}^e(x,y+r/p) = \\ &\frac{T}{\beta}\sum_{i=0}^{q-1}\widehat{\varphi}\left((2x+i)\frac{T}{\beta},(y+i/2)\frac{2\Omega}{\alpha};\tfrac{1}{2}p\alpha T,p\right)\\ &\times\widehat{\gamma}^*\left((2x+i)\frac{T}{\beta},(y+r/p+i/2)\frac{2\Omega}{\alpha};\tfrac{1}{2}\alpha T,1\right)\end{aligned} \quad (14)$$

in the case that q is even and

$$\begin{aligned}&\overline{a}^o(x,y+r/2p) = \\ &\frac{T}{\beta}\sum_{i=0}^{2q-1}\tilde{\varphi}\left((2x+i)\frac{T}{\beta},2y\frac{\Omega}{\alpha};p\alpha T\right)\\ &\times\widehat{\gamma}^*\left((2x+i)\frac{T}{\beta},(y+r/2p+i/2)\frac{2\Omega}{\alpha};\tfrac{1}{2}\alpha T,1\right)\end{aligned} \quad (15)$$

in the case that q is odd, where x extends over an interval of length 1 and y over an interval of length $1/p$ and $1/2p$, respectively. Notice that in the latter expression the ordinary Zak transform appears.

The Fourier transform $\overline{a}^e(x,y)$ is completely determined by the p functions $a_r^e(x,y) = \overline{a}^e(x,y+r/p)$ in the case that q is even, and $\overline{a}^o(x,y)$ is completely determined by the $2p$ functions $a_r^o(x,y) = \overline{a}^o(x,y+r/2p)$ in the case that q is odd. The p functions $a_r^e(x,y)$ and the $2p$ functions $a_r^o(x,y)$ can be combined into p- and $2p$-dimensional column vectors of functions

$$\boldsymbol{a}^e = [a_0^e(x,y),a_1^e(x,y),\ldots,a_{p-1}^e(x,y)]^T \quad (16)$$

and

$$\boldsymbol{a}^o = [a_0^o(x,y),a_1^o(x,y),\ldots,a_{2p-1}^o(x,y)]^T, \quad (17)$$

respectively.

The (modified) Zak transforms $\widehat{\varphi}(2x\frac{T}{\beta},y\frac{2\Omega}{\alpha};\frac{1}{2}p\alpha T,p)$ and $\tilde{\varphi}(2x\frac{T}{\beta},2y\frac{\Omega}{\alpha};p\alpha T)$ are completely determined by the q functions

$$\varphi_i^e(x,y) = \widehat{\varphi}\left((2x+i)\frac{T}{\beta},(y+i/2)\frac{2\Omega}{\alpha};\tfrac{1}{2}p\alpha T,p\right) \quad (18)$$

and the $2q$ functions

$$\varphi_i^o(x,y) = \tilde{\varphi}\left((2x+i)\frac{T}{\beta},(y+i/2)\frac{2\Omega}{\alpha};p\alpha T\right), \quad (19)$$

respectively.

The q and $2q$ functions $\varphi_i^e(x,y)$ and $\varphi_i^o(x,y)$ can be combined into q- and $2q$-dimensional column vectors of functions

$$\boldsymbol{\varphi}^e = [\varphi_0^e(x,y),\varphi_1^e(x,y),\ldots,\varphi_{q-1}^e(x,y)]^T \quad (20)$$

and

$$\boldsymbol{\varphi}^o = [\varphi_0^o(x,y),\varphi_1^o(x,y),\ldots,\varphi_{2q-1}^o(x,y)]^T, \quad (21)$$

respectively.

With the help of these vectors and matrices, Eqs. (14) and (15) can now be expressed in the elegant matrix-vector products

$$\boldsymbol{a}^e = \frac{T}{\beta}\boldsymbol{\Gamma}^{e*}\boldsymbol{\varphi}^e \quad (22)$$

and

$$\boldsymbol{a}^o = \frac{T}{\beta}\boldsymbol{\Gamma}^{o*}\boldsymbol{\varphi}^o. \quad (23)$$

The relations (12) and (13), applied to the arbitrary vectors $\boldsymbol{\varphi}^e$ and $\boldsymbol{\varphi}^o$, respectively, lead to the conditions

$$\boldsymbol{G}^e\boldsymbol{\Gamma}^{e*}\boldsymbol{\varphi}^e = \frac{p\beta}{T}\boldsymbol{\varphi}^e \quad (24)$$

and

$$\boldsymbol{G}^o\boldsymbol{\Gamma}^{o*}\boldsymbol{\varphi}^o = \frac{2p\beta}{T}\boldsymbol{\varphi}^o. \quad (25)$$

Substitution of (22) into (24) yields

$$\boldsymbol{\varphi}^e = \frac{1}{p}\boldsymbol{G}^e\boldsymbol{a}^e, \quad (26)$$

and substitution of (23) into (25) yields

$$\boldsymbol{\varphi}^o = \frac{1}{2p}\boldsymbol{G}^o\boldsymbol{a}^o. \quad (27)$$

Note that (22) represents p equations and q unknowns, (23) represents $2p$ equations and $2q$ unknowns, (26) represents q equations and p unknowns and (27) represents $2q$ equations and $2p$ unknowns. In the case of oversampling ($p > q \geq 1$) the latter two sets of equations are thus underdetermined.

4. EXAMPLE

In this section we determine some dual windows $\gamma(t)$ for the given Gaussian window $g(t) = 2^{1/4}T^{-1/2}e^{-\pi(t/T)^2}$ with $||g|| = 1$, and compare the dual windows for the quincunx case with the dual windows for the rectangular case. As a measure we take the L_2 norm of the difference of the dual window $\gamma(t)$ and the optimum dual window $\gamma_{opt}(t)$, which is proportional to the window $g(t)$, thus we determine $||\gamma - cg||$. One can show that this norm has a minimum if $c = \frac{1}{2}\alpha\beta/||g||^2$ in the case of the quincunx lattice and if $c = \alpha\beta/||g||^2$ in the case of the rectangular lattice.

For this Gaussian window $g(t)$ it is shown that the optimal choice is $\alpha = \beta$ for the rectangular lattice [5]. It is not difficult to show that the optimal choice is $\alpha = \sqrt{3}\beta$ for the quincunx lattice.

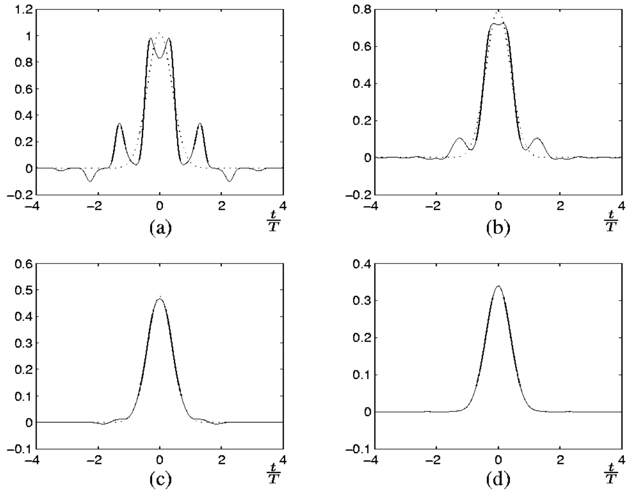

Figure 2: The dual windows (solid line) of a Gaussian elementary signal $g(t) = 2^{\frac{1}{4}} T^{-\frac{1}{2}} \exp(-\pi(t/T)^2)$ and the optimum windows $\gamma_{opt}(t)$ (dotted line) for different values of oversampling, and the difference of the dual window and the optimum dual window in the L_2 norm sense in the case of the quincunx lattice:

(a) $\alpha = \sqrt{\frac{12\sqrt{3}}{7}}, \beta = \frac{1}{3}\sqrt{3}\alpha, p/q = 7/6, ||\gamma - \gamma_{opt}|| = 0.3191$
(b) $\alpha = \sqrt{\frac{4\sqrt{3}}{3}},\ \beta = \frac{1}{3}\sqrt{3}\alpha, p/q = 3/2, ||\gamma - \gamma_{opt}|| = 0.1092$
(c) $\alpha = \sqrt{\frac{4\sqrt{3}}{5}},\ \beta = \frac{1}{3}\sqrt{3}\alpha, p/q = 5/2, ||\gamma - \gamma_{opt}|| = 0.0105$
(d) $\alpha = \sqrt{\frac{4\sqrt{3}}{7}},\ \beta = \frac{1}{3}\sqrt{3}\alpha, p/q = 7/2, ||\gamma - \gamma_{opt}|| = 0.0012$

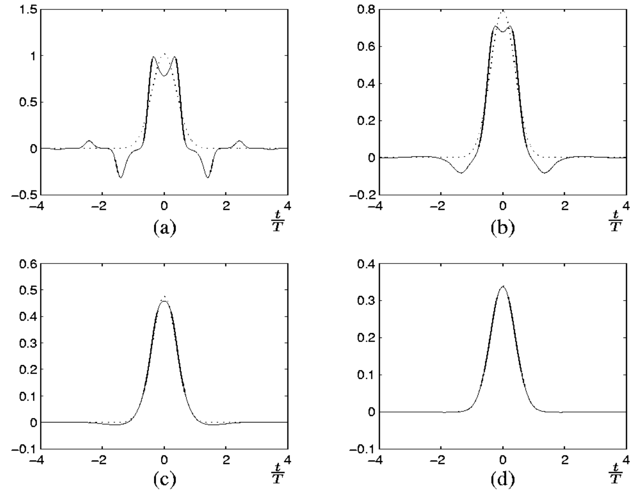

Figure 3: The dual windows (solid line) of a Gaussian elementary signal $g(t) = 2^{\frac{1}{4}} T^{-\frac{1}{2}} \exp(-\pi(t/T)^2)$ and the optimum windows $\gamma_{opt}(t)$ (dotted line) for different values of oversampling, and the difference of the dual window and the optimum dual window in the L_2 norm sense in the case of the rectangular lattice:

(a) $\alpha = \beta = \sqrt{6/7}, p/q = 7/6, ||\gamma - \gamma_{opt}|| = 0.3415$
(b) $\alpha = \beta = \sqrt{2/3}, p/q = 3/2, ||\gamma - \gamma_{opt}|| = 0.1299$
(c) $\alpha = \beta = \sqrt{2/5}, p/q = 5/2, ||\gamma - \gamma_{opt}|| = 0.0158$
(d) $\alpha = \beta = \sqrt{2/7}, p/q = 7/2, ||\gamma - \gamma_{opt}|| = 0.0023$

In Fig. 2 we have depicted the dual windows of $g(t)$ and the optimum dual windows $\gamma_{opt}(t)$ for several choices of α and β for the quincunx lattice. In Fig 3 we have depicted the same for the rectangular lattice.

From this example we can conclude that the dual windows for the quincunx lattice for different values of oversampling are better in the sense that the dual windows resemble better the optimal dual windows in the L_2 sense for this Gaussian window $g(t)$.

5. CONCLUSIONS

We presented Gabor's signal expansion on a quincunx lattice and its relation with a modified Zak transform. It is shown that the modified Zak transform can be used to determine Gabor's expansion coefficients and to find the dual window.

In some cases the Gabor's signal expansion on a quincunx lattice is better in the sense that the dual window resembles better the optimal dual window in the L_2 sense. This is demonstrated for a Gaussian window.

6. REFERENCES

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