PERFORMANCE ANALYSIS OF TWO APPROACHES TO CLOSED LOOP SYSTEM IDENTIFICATION VIA CYCLIC SPECTRAL ANALYSIS

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ABSTRACT

The problem of closed loop system identification given noisy time-domain input-output measurements is considered. It is assumed that the various disturbances affecting the system are zero-mean stationary whereas the closed loop system operates under an external cyclostationary input which is not measured. Noisy measurements of the (direct) input and output of the plant are assumed to be available. The closed loop system must be stable but it is allowed to be unstable in open loop. Recently we proposed two identification algorithms using cyclic-spectral analysis of noisy input-output data. In this paper we provide an asymptotic performance analysis of the recently proposed parameter estimators. Computer simulation examples are presented in support of the analysis.

1. INTRODUCTION

Consider the *'true'* linear system denoted by $\mathcal{S}$

$$\mathcal{S}: \; y(t) = H(q^{-1})u(t) + e(t) = \sum_{i=1}^{\infty} h(i)u(t-i) + e(t), \quad (1)$$

where t is discrete time, q^{-1} is the unit delay operator (i.e. $q^{-1}u(t) = u(t-1)$), $y(t)$ is the noisy output, $u(t)$ is the measured input, and $e(t)$ is the stochastic disturbance. The input $u(t)$ is determined through linear feedback as

$$u(t) = v(t) - F(q^{-1})y(t) = v(t) - \sum_{i=0}^{\infty} f(i)y(t-i) \qquad (2)$$

where $F(q^{-1})$ is the controller transfer function and $v(t)$ is an external input signal (see Fig. 1).

Given an input-output record $\{y(t), u(t), \quad t = 1, 2, ..., T\}$, but the underlying true system $H(q^{-1})$ *unknown*, it of much interest in control, communications and signal processing applications to fit a rational transfer function model parametrized by θ

$$G(q^{-1};\theta) = \frac{B(q^{-1};\theta)}{A(q^{-1};\theta)} = \frac{\sum_{i=1}^{n_b} b_i q^{-i}}{1 + \sum_{i=1}^{n_a} a_i q^{-i}}, \qquad (3)$$

$$\theta = [a_1, a_2, \cdots, a_{n_a}, b_1, b_2, \cdots, b_{n_b}]^T, \qquad (4)$$

to given input-output record. A wide variety of approaches exist [2], [3], [5].

In the presence of the feedback and noise $e(t)$, the input $\{u(t)\}$ is correlated with the output $\{y(t)\}$ so that the standard spectral analysis and related approaches yield biased estimators of the system transfer function and related parameters. For further details, see [2], [3] and [5]. In [2] a nonparametric approach using cyclostationary and/or non-Gaussian inputs was presented to solve this problem. Ref. [2] requires the open loop transfer function to be stable and the approach presented therein is nonparametric. In [6] and [7] we focused on second-order cyclostationarity and parametric approaches and *unlike* [2], *allowed the open loop transfer function* $H(e^{-j\omega})$ *to be unstable*. In [6] and [7] two identification algorithms using cyclic-spectral analysis of noisy input-output data were investigated. In this paper we provide an asymptotic performance analysis of the parameter estimators of [6] and [7].

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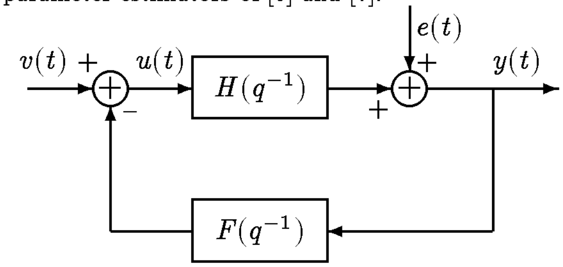

Figure 1. Closed-loop system description

2. PRELIMINARIES

Let $\{u(t)\}$ be a zero-mean second-order almost cyclostationary process, i.e. its second-order cumulant function $c_{uu}(t;\tau) := cum\{u(t+\tau), u(t)\} = E\{u(t+\tau)u(t)\}$ is an almost periodic function in t [1]. Assume that $c_{uu}(t;\tau)$ admits a Fourier series representation w.r.t. t. Then $c_{uu}(t;\tau) := \sum_{\alpha\in A_{uu}} C_{uu}(\alpha;\tau)e^{j\alpha t}$ and $C_{uu}(\alpha;\tau) := \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} c_{uu}(t;\tau)e^{-j\alpha t}$ where $A_{uu} := \{\alpha : C_{uu}(\alpha;\tau) \not\equiv 0, 0 \le \alpha < 2\pi\}$. The Fourier coefficient $C_{uu}(\alpha;\tau)$ is called the second-order cyclic cumulant at cycle frequency α. The set A_{uu} is the countable set of cycle frequencies of the second-order cyclic cumulant of $\{u(t)\}$. The cyclic cumulant spectrum of $\{u(t)\}$ is defined as

$$\mathcal{S}_{uu}(\alpha;\omega) := \sum_{\tau=-\infty}^{\infty} C_{uu}(\alpha;\tau)e^{-j\omega\tau} \qquad (5)$$

Consider the second-order cross-cumulant function $c_{yu}(t,\tau) := cum\{y(t+\tau), u(t)\} = E\{y(t+\tau)u(t)\}$. Then, mimicking the definition of $C_{uu}(\alpha;\tau)$, the cyclic cross-cumulant is defined as $C_{yu}(\alpha;\tau) := \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} c_{yu}(t;\tau)e^{-j\alpha t}$ and the cyclic cross-spectrum $\mathcal{S}_{yu}(\alpha;\omega)$ is defined as

$$\mathcal{S}_{yu}(\alpha;\omega) := \sum_{\tau=-\infty}^{\infty} C_{yu}(\alpha;\tau)e^{-j\omega\tau}. \qquad (6)$$

3. MODEL ASSUMPTIONS

As in [6] and [7], assume the following:

AM1. $\lim_{q\to\infty} H(q^{-1}) = 0$.

AM2. $[1 + H(q^{-1})F(q^{-1})]^{-1}$ is asymptotically stable.

AM3. Disturbance $\{e(t)\}$ is zero-mean stationary. External input $\{v(t)\}$ is zero-mean almost cyclostationary sequence with cycle frequency set A_{vv}.

AM4. For some $\alpha \in A_{vv}$, $|\mathcal{S}_{uu}(\alpha;\omega)| > 0$ for almost all $\omega \in [0,\pi]$ if the proposed approaches utilize the entire frequency range $[0,\pi]$. If only finite number of frequencies ($\geq \frac{n_a+n_b}{2}$) are used then $|\mathcal{S}_{uu}(\alpha;\omega)|$ need be non-zero only for this frequency set.

AM5. Let $x_i(t) \in \{y(t), u(t), v(t), e(t)\}$ for $i = 1, 2, \cdots k$. Let $\underline{\tau}_{(k-1)} := [\tau_1, \ldots, \tau_{k-1}]^T$. Let $c\mathbf{x}(t; \underline{\tau}_{(k-1)}) := \text{cum}\{x_1(t), x_2(t+\tau_1), \ldots, x_k(t+\tau_{k-1})\}$ denote the the kth-order joint cumulant function of random variables $x_i(t)$. Let $C\mathbf{x}(\alpha; \underline{\tau}_{(k-1)}) := \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} c\mathbf{x}(t; \underline{\tau}_{(k-1)}) e^{-j\alpha t}$. The following summability conditions hold true for each $j = 1, \ldots, k-1$ and each $k = 2, 3, \ldots$:

$$\sum_{\tau_1,\ldots,\tau_{k-1}=-\infty}^{\infty} \sup_t [1 + |\tau_j|] |c\mathbf{x}(t; \underline{\tau}_{(k-1)})| < \infty$$

$$\sum_{\tau_1,\ldots,\tau_{k-1}=-\infty}^{\infty} \sup_\alpha [1 + |\tau_j|] |C\mathbf{x}(\alpha; \underline{\tau}_{(k-1)})| < \infty$$

4. TRANSFER FUNCTION ESTIMATOR AND ITS STATISTICS

It has been shown in [6] and [7] that under **AM2-AM4**, we have

$$H(e^{-j\omega}) = \mathcal{S}_{yu}(\alpha;\omega)\mathcal{S}_{uu}^{-1}(\alpha;\omega). \tag{7}$$

The approaches [6] and [7] consist of two steps. First obtain a consistent estimator $\widehat{H}_T(e^{-j\omega};\alpha)$ of $H(e^{-j\omega})$ via consistent estimators $\widehat{\mathcal{S}}_{yu}^{(T)}(\alpha;\omega)$ and $\widehat{\mathcal{S}}_{uu}^{(T)}(\alpha;\omega)$ of $\mathcal{S}_{yu}(\alpha;\omega)$ and $\mathcal{S}_{uu}(\alpha;\omega)$, respectively, based upon the input-output record $\{u(t), y(t), \quad t = 1, 2, \ldots, T\}$. Next estimate the system parameters using the estimated transfer function at various frequencies as '*data*;' this part follows [9]. We will consider estimates based on only a single cyclic frequency $\alpha \in A_{vv}$.

Let $Y^{(T)}(\widetilde{\omega}_k)$ denote the DFT of $\{y(t)\}_{t=1}^T$: $Y^{(T)}(\widetilde{\omega}_k) = \sum_{t=0}^{T-1} y(t+1)e^{-j\widetilde{\omega}_k t}$ where $\widetilde{\omega}_k = \frac{2\pi k}{T}$, $k = 0, 1, \ldots, T-1$. Similarly defined $U^{(T)}(\widetilde{\omega}_k)$. Given the above DFT's, following [1] we define the cross- and auto- cyclic spectrum estimators as

$$\widehat{\mathcal{S}}_{yu}^{(T)}(\alpha;\widetilde{\omega}_k) = \sum_{s=-m_T}^{m_T} \frac{Y^{(T)}(\widetilde{\omega}_{k-s})U^{(T)}(\alpha - \widetilde{\omega}_{k-s})}{T(2m_T+1)} \tag{8}$$

$$\widehat{\mathcal{S}}_{uu}^{(T)}(\alpha;\widetilde{\omega}_k) = \sum_{s=-m_T}^{m_T} \frac{U^{(T)}(\widetilde{\omega}_{k-s})U^{(T)}(\alpha - \widetilde{\omega}_{k-s})}{T(2m_T+1)}. \tag{9}$$

For an arbitrary $\lambda \in [0, 2\pi]$, we define $\widehat{\mathcal{S}}_{yu}^{(T)}(\alpha;\lambda) := \widehat{\mathcal{S}}_{yu}^{(T)}(\alpha;\widetilde{\omega}_k)$ where k is an integer such that $\left|\lambda - \frac{2\pi k}{T}\right|$ is the least. Similarly define $\widehat{\mathcal{S}}_{uu}^{(T)}(\alpha;\lambda)$. In light of (8) and (9) define a coarser frequency grid :

$$\omega_l = \frac{2\pi l}{L_T} + \frac{2\pi(m_T+1)}{T}, \tag{10}$$

with $l = 0, 1, \ldots, L_T - 1$ and $L_T = \lfloor \frac{T}{2m_T+1} \rfloor$.

Lemma 1. [4] Let a sequence of scalar parameters m_T be such that as $T \to \infty$, we have $m_T \to 0$ and $m_T T \to \infty$. Let $k(T)$ with $T = 1, 2, \ldots$ be a sequence of integers such that $\lim_{T\to\infty} \frac{2\pi k(T)}{T} = \lambda$, a fixed frequency. Then under **AM1-AM5**, $\lim_{T\to\infty} E\{\widehat{\mathcal{S}}_{yu}^{(T)}(\alpha;\lambda)\} = \mathcal{S}_{yu}(\alpha;\lambda)$ and $\text{var}\left(\widehat{\mathcal{S}}_{yu}^{(T)}(\alpha;\lambda)\right) = O(\kappa_T^{-1})$ where convergence is uniform in λ, $\text{var}(x) := E\{|x|^2\} - |E\{x\}|^2$ and $\kappa_T = 2m_T + 1$. Consider a fixed set of L distinct frequencies $\{\lambda_n\}_{n=1}^L$ such that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_L < \pi$. Then $\{\widehat{\mathcal{S}}_{yu}^{(T)}(\alpha;\lambda_n)\}_{n=1}^L$ and $\{\widehat{\mathcal{S}}_{uu}^{(T)}(\alpha;\lambda_n)\}_{n=1}^L$ are asymptotically jointly Gaussian random variables. Let $\alpha_1, \alpha_2 \in A_{vv}$ be cycle frequencies that satisfy **AM5**. Then

$$\lim_{T\to\infty} \kappa_T cov\{\widehat{\mathcal{S}}_{x_1 u}^{(T)}(\alpha_1;\lambda_m), \widehat{\mathcal{S}}_{x_2 u}^{(T)}(\alpha_2;\lambda_n)\} =$$

$$\mathcal{S}_{x_1 x_2}(\lambda_m \ominus \lambda_n; \lambda_m)\mathcal{S}_{uu}(\alpha_1 \ominus \lambda_m \oplus \lambda_n \ominus \alpha_2; \alpha_1 - \lambda_m)$$
$$+ \mathcal{S}_{x_1 u}(\lambda_m \oplus \lambda_n \ominus \alpha_2; \lambda_m)\mathcal{S}_{x_2 u}(\alpha_1 \ominus \lambda_m \ominus \lambda_n; -\lambda_n), \tag{11}$$

$$\lim_{T\to\infty} \kappa_T cov\{\widehat{\mathcal{S}}_{x_1 u}^{(T)}(\alpha_1;\lambda_m), \widehat{\mathcal{S}}_{x_2 u}^{(T)*}(\alpha_2;\lambda_n)\} =$$

$$\mathcal{S}_{x_1 x_2}(\lambda_m \oplus \lambda_n; \lambda_m)\mathcal{S}_{uu}(\alpha_1 \ominus \lambda_m \ominus \lambda_n \oplus \alpha_2; \alpha_1 - \lambda_m)$$
$$+ \mathcal{S}_{x_1 u}(\lambda_m \ominus \lambda_n \oplus \alpha_2; \lambda_m)\mathcal{S}_{x_2 u}(\alpha_1 \ominus \lambda_m \oplus \lambda_n; \lambda_n), \tag{12}$$

where $\oplus$ and $\ominus$ denote plus and minus modulo 2π (circular) operations, respectively. □

Clearly, Lemma 1 holds true when we replace y with u in $\widehat{\mathcal{S}}_{yu}^{(T)}(\alpha;\lambda)$. Using the estimated cyclic spectra we have an estimator of the system transfer function at frequency λ

$$\widehat{H}_T(e^{-j\lambda};\alpha) := \widehat{\mathcal{S}}_{yu}^{(T)}(\alpha;\lambda)[\widehat{\mathcal{S}}_{uu}^{(T)}(\alpha;\lambda)]^{-1} \tag{13}$$

provide that $(\widehat{\mathcal{S}}_{uu}^{(T)}(\alpha;\lambda))^{-1}$ exists. It has been shown in [6] and [7] that (i.p.= in probability)

$$\lim_{T\to\infty} \widehat{H}_T(e^{-j\lambda};\alpha) = H(e^{-j\lambda}) \text{ i.p.} \tag{14}$$

Convergence in (14) is uniform in $\lambda \in [0, 2\pi]$.

Remark 1. In the rest of this paper we use ω_l to denote a frequency on the coarse grid (10) and use λ_n to denote a fixed frequency independent of the record length T.

Theorem 1. Suppose that **AM1-AM5** hold true. Let $\alpha_0 \in A_{vv}$ be a cycle frequency that satisfies **AM5**. Consider $\left\{\sqrt{\kappa_T}\left(\widehat{H}_T(e^{-j\lambda_n};\alpha_0) - H(e^{-j\lambda_n})\right)\right\}_{n=1}^L$ where $\{\lambda_n\}_{n=1}^L$ is a fixed set of L distinct frequencies such that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_L < \pi$.

(**A**). If $2\lambda_n \notin A_{vv}$, then $\sqrt{\kappa_T}\left(\widehat{H}_T(e^{-j\lambda_n};\alpha_0) - H(e^{-j\lambda_n})\right)$ converges in distribution to a zero-mean complex circularly symmetric Gaussian random variable with variance given by

$$\lim_{T\to\infty} \kappa_T\, cov\{\widehat{H}_T(e^{-j\lambda_n};\alpha_0), \widehat{H}_T(e^{-j\lambda_n};\alpha_0)\} = \sigma^2(\lambda_n;\alpha_0),$$

$$\lim_{T\to\infty} \kappa_T\, cov\{\widehat{H}_T(e^{-j\lambda_n};\alpha_0), \widehat{H}_T^*(e^{-j\lambda_n};\alpha_0)\} = 0,$$

where $\text{cov}\{x_i, x_j\} = E\{x_i x_j^*\} - E\{x_i\}E\{x_j^*\}$ and

$$\sigma^2(\lambda_n;\alpha_0) = \frac{\mathcal{S}_{uu}^{(s)}(\alpha_0 - \lambda_n)}{|\mathcal{S}_{uu}(\alpha_0;\lambda_n)|^2}\left[|H(e^{-j\lambda_n})|^2\mathcal{S}_{uu}^{(s)}(\lambda_n)\right.$$
$$\left. + \mathcal{S}_{yy}^{(s)}(\lambda_n) - 2Re\{H^*(e^{-j\lambda_n})\mathcal{S}_{yu}^{(s)}(\lambda_n)\}\right]. \tag{15}$$

(**B**). For all $\alpha \in A_{vv}$, define $\tilde{\alpha}(T) := \frac{\alpha T}{2\pi(2m_T+1)}$. Let m_T be chosen so that $\frac{1}{2m_T+1} \notin \{\tilde{\alpha}(T) - \lfloor\tilde{\alpha}(T)\rfloor, \tilde{\alpha}(T) + \tilde{\alpha}_0(T) - \lfloor\tilde{\alpha}(T) + \tilde{\alpha}_0(T)\rfloor, |\tilde{\alpha}(T) - \tilde{\alpha}_0(T)| - \lfloor|\tilde{\alpha}(T) - \tilde{\alpha}_0(T)|\rfloor\}$ for all $\alpha \in A_{vv}$, and either $|\tilde{\alpha}(T) - \tilde{\alpha}_0(T)| - \lfloor|\tilde{\alpha}(T) - \tilde{\alpha}_0(T)|\rfloor$ or $\tilde{\alpha}(T) + \tilde{\alpha}_0(T) - \lfloor\tilde{\alpha}(T) + \tilde{\alpha}_0(T)\rfloor \notin \{0, \frac{T}{2m_T+1} - \lfloor\frac{T}{2m_T+1}\rfloor\}$ for all $\alpha \in A_{vv} - \{\alpha_0\}$. If the set $\{\lambda_n\}_{n=1}^L$

is chosen from the coarse frequency grid (10), then $\left\{\sqrt{\kappa_T}\left(\widehat{H}_T(e^{-j\lambda_n};\alpha_0)-H(e^{-j\lambda_n})\right)\right\}_{n=1}^{L}$ converge in distribution to a zero-mean complex circularly symmetric Gaussian random vector with covariance structure given by

$$\lim_{T\to\infty}\kappa_T\, cov\{\widehat{H}_T(e^{-j\lambda_m};\alpha_0),\widehat{H}_T(e^{-j\lambda_n};\alpha_0)\}$$
$$=\sigma^2(\lambda_n;\alpha_0)\delta_{mn},$$
$$\lim_{T\to\infty}\kappa_T cov\{\widehat{H}_T(e^{-j\lambda_m};\alpha_0),\widehat{H}^*_T(e^{-j\lambda_n};\alpha_0)\}=0.$$

Proof: A proof of part (**A**) appears in [7]. The proof of part (**B**) is more involved and may be found in [8]. (In (15) $\mathcal{S}_{yu}^{(s)}(\lambda)=\mathcal{S}_{yu}(0;\lambda)$). □

Remark 2. If the choice of m_T does not satisfy the requirements of Theorem 1(**B**), then $\widehat{H}_T(e^{-j\lambda_m};\alpha_0)$ will be correlated for certain pairs of frequencies. Extensive simulations suggest that the variance expressions given in Sec. 6 later can be used in this case also (because such lack of independence occurs over a very small subset of all frequency pairs) [8].

5. TWO PARAMETER ESTIMATORS

We will assume that the true model generating the data is in the model set (i.e. $H(e^{-j\omega})$ is of the type $G(e^{-j\omega};\theta)$). Let n_{a0}, n_{b0} and θ_0 denote the true values of n_a, n_b and θ, respectively, such that $G(e^{-j\omega};\theta_0)=H(e^{-j\omega})$ for some θ_0. The fitted model parameters are governed by n_a, n_b and θ, whereas the data are generated by the true model with parameters governed by n_{a0}, n_{b0} and θ_0.

5.1. An Equation Error Formulation

Define

$$\widehat{\theta}_{TL}^{(1)} \;=\; \arg\left\{\min_{\theta\in\Theta_C} J_{1T}(\theta)\right\} \tag{16}$$

where Θ_C is a (large) compact set such that $\theta_0\in\Theta_C$,

$$J_{1T}(\theta)=\frac{1}{L}\sum_{l=1}^{L}\left|A(e^{-j\lambda_l};\theta)\widehat{H}_T(e^{-j\lambda_l};\alpha)-B(e^{-j\lambda_l};\theta)\right|^2,$$

$0<\lambda_1<\lambda_2<\cdots<\lambda_L<\pi$, $B(e^{-j\lambda_l};\theta)=\sum_{i=1}^{n_b} b_i(\theta)e^{-j\lambda_l i}$ and $A(e^{-j\lambda_l};\theta)=1+\sum_{i=1}^{n_a} a_i(\theta)e^{-j\lambda_l i}$. It has been shown in [6] that under **AM1-AM5**, $n_a\geq n_{a0}$, $n_b\geq n_{b0}$ and $\min(n_a-n_{a0},n_b-n_{b0})=0$ such that $n_a+n_b\leq 2L$, it follows that $\lim_{T\to\infty}\widehat{\theta}_{TL}^{(1)}\overset{\text{i.p.}}{=}\theta_0$.

5.2. A Weighted Least-Squares Formulation

Define

$$\widehat{\theta}_{TL}^{(2)}=\arg\left\{\min_{\theta\in\Theta_C} J_{2T}(\theta)\right\} \tag{17}$$

where Θ_C is a (large) compact set such that $\theta_0\in\Theta_C$,

$$J_{2T}(\theta):=\sum_{l=1}^{L}\frac{\left|\widehat{H}_T(e^{-j\lambda_l};\alpha)-G(e^{-j\lambda_l};\theta)\right|^2}{\widehat{\sigma}_T^2(\lambda_l;\alpha)},$$

$\widehat{\sigma}_T^2(\lambda_l;\alpha)$ denotes (15) with all "unknowns" replaced with their consistent estimators (cf. (8) and (9)). It follows from the discussion of Sec. 4 that under **AM1-AM5**, we have $\lim_{T\to\infty}\widehat{\sigma}_T^2(\lambda_l;\alpha)=\sigma^2(\lambda_l;\alpha)$ i.p. uniformly in $\lambda_l\in[0,\pi]$. It has been shown in [7] that under **AM1-AM5**, $n_a\geq n_{a0}$, $n_b\geq n_{b0}$ and $\min(n_a-n_{a0},n_b-n_{b0})=0$, such that $n_a+n_b\leq 2L$, it follows that $\lim_{T\to\infty}\widehat{\theta}_{TL}^{(2)}\overset{\text{i.p.}}{=}\theta_0$.

6. PERFORMANCE ANALYSIS

We will invoke corresponding results from [9], which apply by virtue of Theorem 1 (assuming m_T is chosen to satisfy part (**B**) of Theorem 1), after some straightforward notational changes. Note that [9] which deals with spectral analysis based approaches for open loop systems. But once we reduce the time-domain data to consistent transfer function estimates obeying Theorem 1, the analysis and the results of [9] apply since [9] also works with transfer function estimates that are asymptotically complex (circularly symmetric) Gaussian and are asymptotically independent at distinct frequencies. A difference between [9] and this paper is that the variance expressions for $\widehat{H}_T(e^{-j\lambda_l};\alpha)$ are different in the two papers.

6.1. Equation Error Formulation

It follows from ([9], Sec. IV.C) that under the hypotheses of Theorem 1, $\widehat{\theta}_{TL}^{(1)}$ is asymptotically Gaussian with mean θ_0 and

$$\lim_{T\to\infty}\kappa_T\,\mathrm{cov}\left((\widehat{\theta}_{TL}^{(1)}-\theta_0),(\widehat{\theta}_{TL}^{(1)}-\theta_0)\right)=\frac{1}{L}\mathcal{D}^{-1}\widetilde{\Sigma}_{\theta L}\mathcal{D}^{-1} \tag{18}$$

where $\sigma^2(\lambda_l)$ is given by (15), the symbol $\mathcal{H}$ denotes the conjugate transpose operation,

$$\widetilde{\Sigma}_{\theta L} \;:=\; \frac{1}{L}\sum_{l=1}^{L}\sigma^2(\lambda_l;\alpha)\left(\mathcal{F}_l\mathcal{F}_l^{\mathcal{H}}+\mathcal{F}_l^*\mathcal{F}_l^T\right),$$

$$\mathcal{F}_l \;:=\; A(e^{-j\lambda_l};\theta_0)\overline{\mathcal{C}}_l+\left[A(e^{-j\lambda_l};\theta_0)G(e^{-j\lambda_l};\theta_0)-B(e^{-j\lambda_l};\theta_0)\right]\mathcal{C}_{lg}^*,$$

$$\overline{\mathcal{C}}_l:=\Big[e^{j\lambda_l}G^*(e^{-j\lambda_l};\theta_0)\vdots e^{j2\lambda_l}G^*(e^{-j\lambda_l};\theta_0)\vdots\cdots\vdots e^{jn_a\lambda_l}G^*(e^{-j\lambda_l};\theta_0)\vdots-e^{j\lambda_l}\vdots\cdots\vdots-e^{jn_b\lambda_l}\Big]^T,$$

$$\mathcal{C}_{lg} \;:=\; \left[e^{j\lambda_l}\vdots e^{j2\lambda_l}\vdots\cdots\vdots e^{jn_a\lambda_l}\vdots 0\vdots\cdots\vdots 0\right]^T,$$

$$\mathcal{D}=\frac{1}{L}\sum_{l=1}^{L}\left(\overline{\mathcal{C}}_l\overline{\mathcal{C}}_l^{\mathcal{H}}+\overline{\mathcal{C}}_l^*\overline{\mathcal{C}}_l^T\right)=\text{ independent of }\theta_0.$$

6.2. Weighted Least-Squares

It follows from ([9], Sec. V.C) that under the hypotheses of Theorem 1, $\widehat{\theta}_{TL}^{(2)}$ is asymptotically Gaussian with mean θ_0 and

$$\lim_{T\to\infty}\kappa_T\,\mathrm{cov}\left((\widehat{\theta}_{TL}^{(2)}-\theta_0),(\widehat{\theta}_{TL}^{(2)}-\theta_0)\right)\;=\;\frac{1}{L}\left[\Sigma_{\theta L}^{(2)}\right]^{-1} \tag{19}$$

where $\sigma^2(\lambda_l)$ is given by (15)),

$$\Sigma_{\theta L}^{(2)}=\frac{1}{L}\sum_{l=1}^{L}\frac{\{\mathcal{D}_l(\theta_0)\mathcal{D}_l^{\mathcal{H}}(\theta_0)+\mathcal{D}_l^*(\theta_0)\mathcal{D}_l^T(\theta_0)\}}{|A(e^{-j\lambda_l};\theta_0)|^2\sigma^2(\lambda_l;\alpha)},$$

$$\mathcal{D}_l(\theta):=\Big[e^{j\lambda_l}G^*(e^{-j\lambda_l};\theta)\vdots e^{j2\lambda_l}G^*(e^{-j\lambda_l};\theta)\vdots\cdots\vdots e^{jn_a\lambda_l}G^*(e^{-j\lambda_l};\theta)\vdots-e^{j\lambda_l}\vdots\cdots\vdots-e^{jn_b\lambda_l}\Big]^T.$$

7. SIMULATION EXAMPLE

This example is based upon [10]. The open loop plant is given by

$$H(q^{-1}) = \frac{q^{-1} + 0.5q^{-2}}{1 - 1.85q^{-1} + 0.525q^{-2}}; \quad \text{poles}: \quad 1.5, 0.35.$$

The controller $F(q^{-1})$ is given by $F(q^{-1}) = [0.35 - 0.28q^{-1}][1 - 0.8q^{-1}]^{-1}$. The closed loop system is stable. We take

$$e(t) = \frac{1 - 1.7959q^{-1} + 1.4328q^{-2} - 0.59608q^{-3} + 0.08738q^{-4}}{1 - 1.7q^{-1} + 0.33q^{-2} + 1.063q^{-3} - 0.6408q^{-4}}\epsilon(t)$$

and the cyclostationary external input signal is chosen as $v(t) = cos(\frac{3\pi t}{8})\xi(t)$ where $\epsilon(t)$ and $\xi(t)$ are zero-mean i.i.d. Gaussian random sequences with unit variance, and they are independent of each other. This leads to $A_{vv} = \{0, 0.75\pi, 1.25\pi\}$. We selected the cycle frequency $\alpha = 0.75\pi$ for system identification via cyclic spectral analysis. The power of $\{\epsilon(t)\}$ was scaled to achieve a closed loop output SNR of 10 dB. Let $s(t) =$ contribution of $v(t)$ alone to $y(t)$ and and let $\eta(t) =$ contribution of $e(t)$ alone to $y(t)$. Then output SNR is defined as the ratio

$$\text{SNR} = \frac{\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E[s^2(t)]}{\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E[\eta^2(t)]}.$$

The required cyclic spectra (auto and cross) were estimated via frequency-domain averaging of cyclic periodogram/ cross-periodogram using non-overlapping rectangular windows (see (8) and (9)). Tables 1 and 2 show the results of averages over 100 Monte Carlo runs based upon a record length $T = 2048$ with $2m_{2048} + 1 = 21$ in (8),(9). Tables 3 and 4 show the same for a record length $T = 8192$ with $2m_{8192} + 1 = 91$. We have used two different approaches to compute the asymptotic variances. In the first approach we use the complete knowledge of the true system where smoothed true auto- and cross-cyclic spectra and a smoothed true transfer function as well as θ_0 are used to compute (18) and (19). We denote the asymptotic variance computed in this way by σ_{t1}^2. In the second approach we replace the true system quantities in (18) and (19) by the averages over 100 Monte Carlo runs of estimated smoothed cyclic spectra and estimated parameters. This approach allows us to estimate the asymptotic variances using only the observed input and output data over multiple records. We denote the resulting asymptotic variance by σ_{t2}^2. Note that σ_{t1}^2 uses "asymptotic" expressions whereas σ_{t2}^2 is based on finite record estimates, and therefore incorporates finite record length effects; it turns out to be more accurate for this example.

TABLE 1 : based on 100 Monte Carlo runs
Equation Error Formulation

	T = 2048				
	True	Mean	σ_e	σ_{t1}	σ_{t2}
a_1	−1.850	−1.855	0.054	0.030	0.057
a_2	0.525	0.560	0.054	0.031	0.060
b_1	1.000	0.981	0.066	0.035	0.068
b_2	0.500	0.453	0.062	0.038	0.073

$\sigma_e =$ experimental standard deviation (SD). $\sigma_{t1} =$ asymptotic SD based on (18) or (19) using knowledge of true system in computed the required entities. $\sigma_{t2} =$ asymptotic SD based on Monte Carlo evaluation of entities needed in (18) or (19).

TABLE 2 : based on 100 Monte Carlo runs
Weighted Least-Squares

	T = 2048				
	True	Mean	σ_e	σ_{t1}	σ_{t2}
a_1	−1.850	−1.845	0.023	0.012	0.031
a_2	0.525	0.519	0.027	0.012	0.032
b_1	1.000	0.991	0.031	0.014	0.035
b_2	0.500	0.508	0.033	0.016	0.038

TABLE 3 : based on 100 Monte Carlo runs
Equation Error Formulation

	T = 8192				
	True	Mean	σ_e	σ_{t1}	σ_{t2}
a_1	−1.850	−1.849	0.026	0.016	0.036
a_2	0.525	0.529	0.027	0.016	0.037
b_1	1.000	0.994	0.027	0.018	0.038
b_2	0.500	0.496	0.031	0.020	0.041

TABLE 4 : based on 100 Monte Carlo runs
Weighted Least-Squares

	T = 8192				
	True	Mean	σ_e	σ_{t1}	σ_{t2}
a_1	−1.850	−1.845	0.014	0.006	0.015
a_2	0.525	0.520	0.016	0.006	0.016
b_1	1.000	0.994	0.016	0.007	0.017
b_2	0.500	0.506	0.018	0.008	0.018

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