RECTIFICATION OF CROSS SPECTRAL MATRICES FOR ARRAYS OF ARBITRARY GEOMETRY

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ABSTRACT

In high resolution methods applied to Uniform Linear Arrays (ULA), the pre-processing that consists in forcing the estimated Cross Spectral Matrix (CSM) to be Toeplitz by averaging its elements along its diagonals is known to increase drastically the resolving power: that is why it is always done in practise. However, this approach is limited to linear arrays because of the required Toeplitz structure for the CSM. This paper *generalizes* this technique to arrays of *arbitrary geometry*: the developed method is referred to as *rectification*. It proceeds by searching first for a vector subspace of hermitian matrices that contains the manifold generated by the CSM's when the Angle Of Arrival varies: this preliminary step is performed only one time for a given array geometry. Next, rectification of estimated CSM's is achieved by projecting them onto this subspace, resulting in denoising and increased resolving power of source localization methods at *a very low computational cost*. As a by product, the storage requirements for the CSM's are greatly reduced.

1. INTRODUCTION

In high resolution localization of uncorrelated sources using Uniform Linear Arrays (ULA), the pre-processing that consists in forcing the estimated Cross Spectral Matrix (CSM) to be Toeplitz by averaging its elements along its diagonals is known to increase drastically the resolving power [1], and its theoretical impact on resolution has been studied in [3]. However, this approach is limited to linear arrays because of the required Toeplitz structure for the CSM: therefore, there is a real need for developping similar simple pre-processing for other arrays. One possible approach with arrays of arbitrary geometry could be the mapping of the actual array into a virtual linear array as was done in [2] in the context of spatial smoothing: however, this is an *indirect* way to tackle the problem since it involves first the intermediate problem of designing the appropriate virtual array. Instead, we develop in this paper a *direct* method for arrays of arbitrary geometry that will be referred to as *rectification*. It proceeds by searching first for a vector subspace $\mathcal{E}$ of hermitian matrices that contains the manifold generated by the CSM's when the Angle Of Arrival (AOA) varies: this preliminary step is performed only one time for a given array geometry. Next, rectification of estimated CSM's is achieved by projecting them onto this subspace $\mathcal{E}$, resulting in denoising and increased resolving power of source localization methods as demonstrated by simulations. The paper is organized as follows. Section 2 introduces notations and background in the case of ULA. Section 3 develops the proposed algorithm. Section 4 shows how to determine the dimension of $\mathcal{E}$. Finally, section 5 presents simulation results that demonstrate the increased resolving power when applying localization methods after rectification.

2. NOTATIONS AND BACKGROUND

Let us consider an array of M sensors with steering vector $\mathbf{a}(\theta, f)$ where θ is the AOA and f is frequency. We will assume that the array and the sources are coplanar, and that the steering vectors have been normalized to 1. Let

$$\mathbf{\Gamma}(f) = \sum_{p=1}^{P} \sigma_p(f)\, \mathbf{a}(\theta_p, f)\mathbf{a}^H(\theta_p, f) + \sigma_n(f)\, \mathbf{I} \qquad (1)$$

be the CSM at f, where $\sigma_p(f)$ and $\sigma_n(f)$ denote the sources and noise Power Spectrum Density (PSD). The CSM is conventionally estimated by windowing and Fourier transforming the array output data over N consecutive time intervals, and computing the empirical covariance matrix of the Fourier transformed data $\mathbf{x}_1(f), \cdots, \mathbf{x}_N(f)$:

$$\widehat{\mathbf{\Gamma}}_w(f) = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}_n(f)\, \mathbf{x}_n^H(f). \qquad (2)$$

N will be referred to as the number of data. Provided that the resolution of the Fourier transform is high enough, this estimate is known to be unbiased and complex Wishart distributed with N degrees of freedom. From now on, f will be fixed and the dependence of CSM's, PSD's and steering vectors on f will be omitted for clarty.

For uniform linear arrays and uncorrelated sources, $\mathbf{\Gamma}$ is well known to exhibit an Hermitian Toeplitz structure. This Toeplitz structure is usually taken into account by averaging $\widehat{\mathbf{\Gamma}}_w$ along its diagonals. This simple pre-processing increases the estimation precision of the CSM and thereby improves the resolving power of localization methods. It can be interpreted as the projection of the estimated CSM onto the vector space spanned by Hermitian Toeplitz matrices if the scalar product of two hermitian matrices $\mathbf{A}$ and $\mathbf{B}$ is defined as $tr(\mathbf{A}\,\mathbf{B})$ where $tr(.)$ denotes the trace of a matrix. As we will see later, the proposed method for arbitrary arrays will also project the non parametric CSM estimate (2) onto an appropriate vector subspace of hermitian matrices.

For clarty, let us recall briefly a few elementary facts on hemitian matrices and set some notations that will be used later. Separating real and imaginary parts, the set $\mathcal{H}$ of hermitian matrices of order M is a vector space of dimension M^2 over the field of real numbers. $(\mathbf{A}, \mathbf{B}) = tr(\mathbf{A}\,\mathbf{B})$ defines a scalar product with corresponding norm $||\,.\,||_F$, the so-called Frobenius norm. Let $vec(\mathbf{A})$ be the real vector of dimension $2\,M^2$ whose components are equal to the real and imaginary parts of the elements of $\mathbf{A}$ taken in any (but fixed) order, and denote by $unvec(.)$ the inverse operation: $unvec(vec((\mathbf{A})) = \mathbf{A}$. Then, the scalar product of $\mathbf{A}$ and $\mathbf{B}$ is simply equal to the ordinary scalar product of $vec(\mathbf{A})$ and $vec(\mathbf{B})$: $(\mathbf{A}, \mathbf{B}) = vec(\mathbf{A})^T\, vec(\mathbf{B})$.

3. RECTIFICATION OF CSM'S

From (1), CSM's are linear combinations of the identity matrix and matrices of the form:

$$\mathbf{a}(\theta)\mathbf{a}^H(\theta).$$

We will look for an L dimensional subspace $\mathcal{E}$ of hermitian matrices (the value of L will be discussed later) that contains the identity matrix and which is as close as possible of the manifold generated by the matrices $\mathbf{a}(\theta)\mathbf{a}^H(\theta)$ when the AOA θ varies over the set Θ of allowed values. Generally, it must be emphasized that this subspace $\mathcal{E}$ will not contain exactely the matrices $\mathbf{a}(\theta)\mathbf{a}^H(\theta)$ (except for particular array geometries like the ULA). However, it can be made as close of them as we wish by choosing L sufficiently large: this issue will be discussed in detail in section 4 where a practical rule for determining L will be given.
Choosing a quadratic fit, we first perform the following preliminary step.

Preliminary step. Let $\mathcal{E}$ be any L dimensional subspace of hermitian matrices that contains the identity matrix, and $\Pi[\,]$ the orthogonal projector operator onto $\mathcal{E}$. Consider the following criterion

$$f(\Pi) = \int_\Theta ||\Pi[\mathbf{a}(\theta)\mathbf{a}^H(\theta)] - \mathbf{a}(\theta)\mathbf{a}^H(\theta)||_F^2\, g(\theta)\, d\theta\,, \quad (3)$$

where $g(\theta)$ is a given positive weighting function. Then, the subspace $\mathcal{E}$ that minimizes (3) under the constraint $\mathbf{I} \in \mathcal{E}$ can be obtained as follows. Set

$$\mathbf{d}(\theta) = vec(\mathbf{a}(\theta)\mathbf{a}^H(\theta) - M^{-1}\mathbf{I}) \quad (4)$$

and

$$\mathbf{R} = \int_\Theta \mathbf{d}(\theta)\mathbf{d}(\theta)^T\, g(\theta)\, d\theta\,. \quad (5)$$

Let $\mathbf{u}_1, \cdots, \mathbf{u}_{L-1}$ be the $L-1$ greatest eigenvectors of $\mathbf{R}$. The L matrices $\mathbf{U}_1 = unvec(\mathbf{u}_1), \cdots, \mathbf{U}_{L-1} = unvec(\mathbf{u}_{L-1})$, $\mathbf{U}_L = 1/\sqrt{M}\,\mathbf{I}$ form an othonormal basis of $\mathcal{E}$.

Proof: See appendix A

Remark 1: In practise, two values of the weighting function $g(\theta)$ in (3) will be used: $g(\theta) = 1$ when an analytic expression is available for the steering vectors, and $g(\theta) = \sum_i \delta(\theta - \theta_i)$ when the steering vectors have been measured for a discrete set of AOA θ_i as occurs when the array has been experimentally calibrated.

Rectification scheme. Assume that the value of L has been choosen such that the difference between $\mathbf{a}(\theta)\mathbf{a}^H(\theta)$ and $\Pi[\mathbf{a}(\theta)\mathbf{a}^H(\theta)]$ is negligible. Thus, $\mathcal{E}$ contains any exact CSM, and the rectification of the estimated CSM $\widehat{\mathbf{\Gamma}}_w$ (2) consists in projecting it onto $\mathcal{E}$, thereby increasing the estimation precision:

$$\widehat{\mathbf{\Gamma}} = \sum_{l=1}^{L} tr(\widehat{\mathbf{\Gamma}}_w\, \mathbf{U}_l)\, \mathbf{U}_l\,. \quad (6)$$

Remark 2: In the ULA case, it is easy to show that the averaging along the diagonals of the estimated CSM $\widehat{\mathbf{\Gamma}}_w$ to make it Toeplitz reduces to (6) with $L = 2\,M$ and appropriate Hermitian matrices $\mathbf{U}_l$.

4. DETERMINATION OF THE DIMENSION OF THE APPROXIMATING SUBSPACE $\mathcal{E}$

By projecting the CSM estimate $\widehat{\mathbf{\Gamma}}_w$ onto a subspace of Hermitian matrices, rectification lowers the original variance of the elements of $\widehat{\mathbf{\Gamma}}_w$. However, it can also bias the estimate when $\mathcal{E}$ does not contain exactly the array manifold. We propose to choose the dimension L of $\mathcal{E}$ so that the bias induced by rectifying $\widehat{\mathbf{\Gamma}}_w$ is much smaller than the original MSE of $\widehat{\mathbf{\Gamma}}_w$. As shown below , this results in a simple rule for determining L : it is the smallest value that ensures the following inequality

$$N \sup_{\theta} \| \Pi^{\perp} \left[\mathbf{a}(\theta)\mathbf{a}^H(\theta) \right] \|_F^2 \ll 1 \quad (7)$$

where $\Pi^{\perp}[\ ]$ is the projection operator onto $\mathcal{E}^{\perp}$ and N is the number of data. In practise, the above inequality was considered as satisfied when its left member was less than 0.01. The sequel of this section provides a proof of (7).

MSE of $\widehat{\mathbf{\Gamma}}_w$. Set

$$\widehat{\mathbf{\Gamma}}_w = \mathbf{\Gamma} + \Delta\mathbf{\Gamma}_w. \quad (8)$$

From (2), $\widehat{\mathbf{\Gamma}}_w$ is Wishart distributed. Let

$$\Delta_w = E[\|\Delta\mathbf{\Gamma}_w\|_F^2] \quad (9)$$

denote the MSE for $\widehat{\mathbf{\Gamma}}_w$. Standard statistical properties of Wishart matrices yield:

$$\begin{aligned} \Delta_w &= \frac{1}{N} tr^2(\mathbf{\Gamma}) \\ &= \frac{1}{N} (\sum_p \sigma_p + M\,\sigma_n)^2 \quad (10) \end{aligned}$$

where σ_p and σ_n are defined in equation (1).

Bias of the rectified CSM. Let us denote by $\Pi[\ ]$ and $\Pi^{\perp}[\ ]$ the projection operators onto $\mathcal{E}$ and its orthogonal. The mean of the rectified CSM $\widehat{\mathbf{\Gamma}}$ is given by

$$E[\widehat{\mathbf{\Gamma}}] = E\left[\Pi[\widehat{\mathbf{\Gamma}}_w]\right] = \Pi\left[E[\widehat{\mathbf{\Gamma}}_w]\right] = \Pi[\mathbf{\Gamma}] \ .$$

After substituting for expression (1) of $\mathbf{\Gamma}$ and noting that $\Pi^{\perp}[\mathbf{I}] = 0$, we obtain the bias of the rectified CSM :

$$\mathbf{\Gamma} - E[\widehat{\mathbf{\Gamma}}] = \Pi^{\perp}[\mathbf{\Gamma}] = \sum_{p=1}^{P} \sigma_p \, \Pi^{\perp}\left[\mathbf{a}(\theta_p)\mathbf{a}^H(\theta_p)\right]. \quad (11)$$

To compare the bias to the MSE (10), we take its squared norm:

$$\begin{aligned} \Delta_b &= \|\mathbf{\Gamma} - E[\widehat{\mathbf{\Gamma}}]\|_F^2 \\ &\le \left(\sum_{p=1}^{P} \sigma_p \|\Pi^{\perp}\left[\mathbf{a}(\theta_p)\mathbf{a}^H(\theta_p)\right]\|_F \right)^2 \\ &\le (\sum_{p=1}^{P} \sigma_p)^2 \sup_{\theta} \|\Pi^{\perp}\left[\mathbf{a}(\theta)\mathbf{a}^H(\theta)\right]\|_F^2 \quad (12) \end{aligned}$$

where the upper bound for Δ_b can be reached for a single source with appropriate AOA. Comparison of expressions (12) and (10) shows that:

$$\Delta_b/\Delta_w \le N \sup_{\theta} \|\Pi^{\perp}\left[\mathbf{a}(\theta)\mathbf{a}^H(\theta)\right]\|_F^2 \quad (13)$$

where the upper bound can be reached in the single source case with $\sigma_n = 0$. Thus, inequality (7) ensures as announced that the bias induced by rectifying $\widehat{\mathbf{\Gamma}}_w$ is much smaller than the original MSE of $\widehat{\mathbf{\Gamma}}_w$.

5. SIMULATIONS

We consider an $M = 10$ sensors circular array with ra-

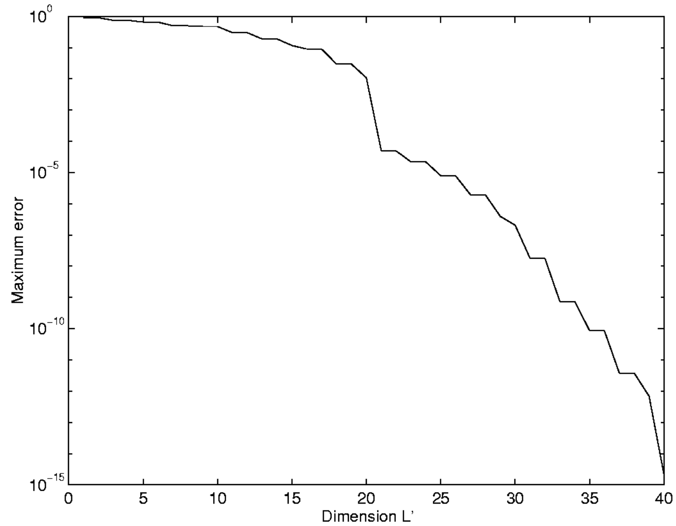

Figure 1: $\sup_{\theta} \|\Pi^{\perp}\left[\mathbf{a}(\theta)\mathbf{a}^H(\theta)\right]\|_F^2$ as a function of L.

dius $0.7\,\lambda$ where λ is the wave-length (greater radii yield vary high side lobes in the array response). Thus, the dimension of the hermitian matrices space is $M^2 = 100$. Figure 1 displays the maximum distance between the matrices $\mathbf{a}(\theta)\mathbf{a}^H(\theta)$ and the subspace $\mathcal{E}$ as a function of $L = \dim\mathcal{E}$. This distance falls suddenly for $L = 21$ where it is equal to $5\,10^{-5}$: according to the selection rule of section 4, this value of L is convenient for a number of data less or equal to 200. So, *only* $L = 21$ *dimensions suffice for representing CSM's with very high accuracy, while the dimension of the hermitian matrices space is* $M^2 = 100$ *!*

The next simulation illustrates the benefits of the proposed rectification scheme. There are $P = 2$ sources, each with SNR per sensor -7 dB. The number of data is $N = 200$. Figure 2 displays the results obtained when applying MUSIC to the original CSM estimates (upper plots) and to the rectified ones using $L = 21$ (lower plots) : improvment is spectacular.

6. CONCLUSION

A preprocessing technique, originally introduced for uniform linear arrays, has been extended in this paper to arrays of arbitrary geometries. For a any array geometry, we

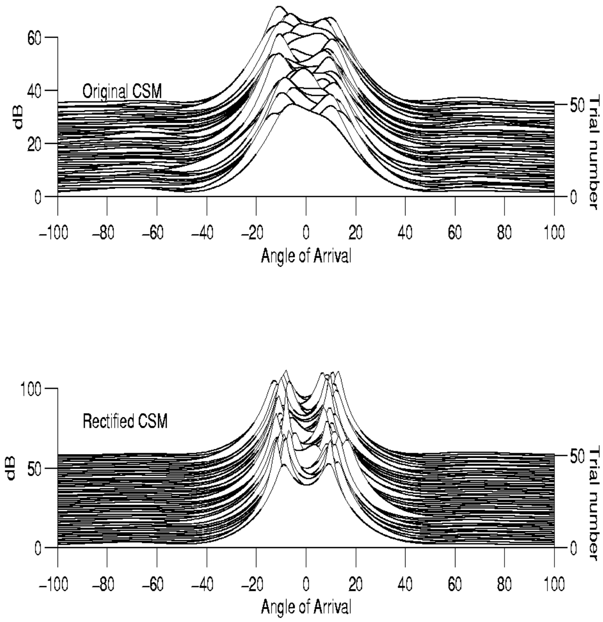

Figure 2: MUSIC applied to: original matrices (upper plot), rectified matrices (lower plot).

have shown how a subspace $\mathcal{E}$ of hermitian matrices can determined so that it contains all the CSM's with a given accuracy. Simply projecting the estimated CSM onto $\mathcal{E}$ improves the estimation precision of the CSM and increases the resolving power of localization methods as confirmed by simulations.

7. REFERENCES

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A. APPENDIX

Let us first decompose the space $\mathcal{E}$ into $\mathcal{E}_1 \oplus \mathcal{E}_2$ where: $\mathcal{E}_2$ is the one dimensional space spanned by $\mathbf{I}$, and $\mathcal{E}_1$ is the $L-1$ dimensional space orthogonal to $\mathcal{E}_2$. Let Π_1 and Π_2 be the corresponding projection operators, so that $\Pi = \Pi_1 + \Pi_2$. Denote by

$$\mathbf{D}(\theta) = \mathbf{a}(\theta)\mathbf{a}^H(\theta) - M^{-1}\mathbf{I} \qquad (14)$$

the residue of the projection of $\mathbf{a}(\theta)\mathbf{a}^H(\theta)$ onto $\mathcal{E}_2$. Then, it easy to check that criterion (3) is equivalent to :

$$f_1(\Pi_1) = \int_\Theta \|\Pi_1[\mathbf{D}(\theta)] - \mathbf{D}(\theta)\|_F^2 \, g(\theta)\, d\theta\,. \qquad (15)$$

Let us denote by $vec(\mathcal{E}_1)$ the subspace of dimension $L-1$ spanned by the vectors $vec(\mathbf{A})$ where $\mathbf{A} \in \mathcal{E}_1$. Let $\mathbf{\Pi}$ be the rank $L-1$ projection matrix onto $vec(\mathcal{E}_1)$. Then, criterion (15) can be rewritten :

$$\begin{aligned} f_1(\mathbf{\Pi}) &= \int_\Theta \|\mathbf{\Pi}\,\mathbf{d}(\theta) - \mathbf{d}(\theta)\|^2 \, g(\theta)\, d\theta \\ &= \int_\Theta \|\mathbf{d}(\theta)\|^2 \, g(\theta)\, d\theta - \int_\Theta \mathbf{d}^T(\theta)\mathbf{\Pi}\,\mathbf{d}(\theta)\, g(\theta)\, d\theta \end{aligned}$$

where $\mathbf{d}(\theta) = vec(\mathbf{D}(\theta))$. Minimizing $f_1(\mathbf{\Pi})$ is clearly equivalent to maximize

$$\begin{aligned} h(\mathbf{\Pi}) &= \int_\Theta \mathbf{d}^T(\theta)\mathbf{\Pi}\,\mathbf{d}(\theta)\, g(\theta)\, d\theta \\ &= tr\left(\mathbf{\Pi} \int_\Theta \mathbf{d}(\theta)\,\mathbf{d}^T(\theta)\, g(\theta)\, d\theta\right) \\ &= tr(\mathbf{\Pi}\,\mathbf{R}) \end{aligned} \qquad (16)$$

over rank $L-1$ projectors. The differential of $h(\mathbf{\Pi})$ must vanish, which yields :

$$\delta h = tr(\delta\mathbf{\Pi}\,\mathbf{R}) = 0\,. \qquad (17)$$

To characterize elementary variations of the projector $\delta\mathbf{\Pi}$, we note that they are obtained by making arbitrary infinitesimal rotations of its invariant subspace. Rotations in the neighbourhood of the identity matrix can be expanded as $\mathbf{I} + \delta\mathbf{\Omega} + \cdots + \delta^n\mathbf{\Omega} + \cdots$, where the elements of $\delta^n\mathbf{\Omega}$ are of order n with respect to those of $\delta\mathbf{\Omega}$. This rotation being unitary, we must have $(\mathbf{I}+\delta\mathbf{\Omega}+\cdots)^2 = \mathbf{I}$ which yields for the first order term:

$$\delta\mathbf{\Omega} + \delta\mathbf{\Omega}^T = 0\,. \qquad (18)$$

Thus, we get :

$$\mathbf{\Pi} + \delta\mathbf{\Pi} = (\mathbf{I} + \delta\mathbf{\Omega} + \cdots)\,\mathbf{\Pi}\,(\mathbf{I} + \delta\mathbf{\Omega} + \cdots)^T\,, \qquad (19)$$

which gives by retaining only the first order term in $\delta\mathbf{\Omega}$:

$$\delta\mathbf{\Pi} = \delta\mathbf{\Omega}\,\mathbf{\Pi} + \mathbf{\Pi}\,\delta\mathbf{\Omega}^T\,, \qquad (20)$$

where $\delta\mathbf{\Omega} = (\delta\Omega_{ij})$ is any matrix satisfying (18). By substituting for $\delta\mathbf{\Pi}$ from (20) into (17) and taking (18) into account, we obtain:

$$\delta h = tr(\delta\mathbf{\Omega}\,[\mathbf{\Pi}\,\mathbf{R} - \mathbf{R}\,\mathbf{\Pi}]) = 2\sum_{i<j} \delta\Omega(i,j)\, S(j,i) \qquad (21)$$

where $\mathbf{S} = \mathbf{\Pi}\,\mathbf{R} - \mathbf{R}\,\mathbf{\Pi}$. Equation (21) must vanish for any $\delta\Omega(i,j)$ which implies $\mathbf{S} = 0$, or equivalently $\mathbf{\Pi}\,\mathbf{R} = \mathbf{R}\,\mathbf{\Pi}$: $\mathbf{\Pi}$ and $\mathbf{R}$ commute. Consequently, they share the same eigenvectors, and to maximize (16) these eigenvectors must be the $L-1$ greatest ones $\mathbf{u}_1, \cdots, \mathbf{u}_{L-1}$ of $\mathbf{R}$. Thus $\mathbf{U}_1 = unvec(\mathbf{u}_1), \cdots, \mathbf{U}_{L-1} = unvec(\mathbf{u}_{L-1})$ form a basis of $\mathcal{E}_1$ and an orthonormal basis of $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is simply obtained by adding $\mathbf{U}_L = 1/\sqrt{M}\,\mathbf{I}$ to it.